

SPHERICAL HARMONICS IN p DIMENSIONS

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Preface

The authors prepared the following booklet in order to make several useful topics from the theory of special functions, in particular the spherical harmonics and Legendre polynomials of \mathbb{R}^p , available to undergraduates studying physics or mathematics. With this audience in mind, nearly all details of the calculations and proofs are written out, and extensive background material is covered before beginning the main subject matter. The reader is assumed to have knowledge of multivariable calculus and linear algebra (especially inner product spaces) as well as some level of comfort with reading proofs.

Literature in this area is scant, and for the undergraduate it is virtually nonexistent. To find the development of the spherical harmonics that arise in \mathbb{R}^3 , physics students can look in almost any text on mathematical methods, electrodynamics, or quantum mechanics (see [1], [2], [6], [8], [11], for example), and math students can search any book on boundary value problems, PDEs, or special functions (see [3], [13], for example). However, the undergraduate will have a very difficult time finding accessible material on the corresponding topics in arbitrary \mathbb{R}^p .

The authors used Hochstadt's *The Functions of Mathematical Physics* [5] as a primary reference (which is, unfortunately, out of print). If the reader seeks a much more concise treatment of spherical harmonics in an arbitrary number of dimensions written at a higher level, [5] is recommended. Much of the theory developed below can be found there.

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Chapter 1

Introduction and Motivation

Many important equations in physics involve the *Laplace operator*, which is given by

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (1.1)$$

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad (1.2)$$

in two and three dimensions¹, respectively. We will see later (Proposition 2.1) that the Laplace operator is invariant under a rotation of the coordinate system. Thus, it arises in many physical situations in which there exists spherical symmetry, i.e., where physical quantities depend only on the radial distance r from some center of symmetry \mathcal{O} . For example, the electric potential V in free space is found by solving the *Laplace equation*,

$$\Delta\Phi = 0, \quad (1.3)$$

which is rotationally invariant. Also, in quantum mechanics, the wave function ψ of a particle in a central field can be found by solving the *time-independent Schrödinger equation*,

$$\left[-\frac{\hbar^2}{2m}\Delta + V(r) \right] \psi = E\psi \quad (1.4)$$

where \hbar is Planck's constant, m is the mass of the particle, $V(r)$ is its potential energy, and E is its total energy.

We will give a brief introduction to these problems in two and three dimensions to motivate the main subject of this discussion. In doing this, we will get a preview of some of the properties of *spherical harmonics* — which, for now, we can just think of as some special set of functions — that we will develop later in the general setting of \mathbb{R}^p .

¹We may drop the subscript if we want to keep the number of dimensions arbitrary.

1.1 Separation of Variables

Two-Dimensional Case

Since we are interested in problems with spherical symmetry, let us rewrite the Laplace operator in spherical coordinates, which in \mathbb{R}^2 are just the ordinary polar coordinates²,

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right). \quad (1.5)$$

Alternatively,

$$x = r \cos \phi, \quad y = r \sin \phi.$$

Using the chain rule, we can rewrite the Laplace operator as

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}. \quad (1.6)$$

In checking this result, perhaps it is easiest to begin with (1.6) and recover (1.1). First we compute

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}, \quad (1.7)$$

which implies that

$$\frac{\partial^2}{\partial r^2} = \cos^2 \phi \frac{\partial^2}{\partial x^2} + 2 \sin \phi \cos \phi \frac{\partial^2}{\partial x \partial y} + \sin^2 \phi \frac{\partial^2}{\partial y^2},$$

and

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} = -r \sin \phi \frac{\partial}{\partial x} + r \cos \phi \frac{\partial}{\partial y},$$

which gives

$$\frac{\partial^2}{\partial \phi^2} = r^2 \sin^2 \phi \frac{\partial^2}{\partial x^2} - 2r^2 \sin \phi \cos \phi \frac{\partial^2}{\partial x \partial y} + r^2 \cos^2 \phi \frac{\partial^2}{\partial y^2}.$$

Inserting these into (1.6) gives us back (1.1).

Thus, (1.3) becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

²The polar angle ϕ actually requires a more elaborate definition, since \tan^{-1} only produces angles in the first and fourth quadrants. However, this detail will not concern us here.

To solve this equation it is standard to assume that $\Phi(r, \phi) = \chi(r)Y(\phi)$, where $\chi(r)$ is a function of r alone and $Y(\phi)$ is a function of ϕ alone. Then

$$Y \frac{d^2\chi}{dr^2} + \frac{Y}{r} \frac{d\chi}{dr} + \frac{\chi}{r^2} \frac{d^2Y}{d\phi^2} = 0.$$

Multiplying by $r^2/\chi Y$ and rearranging,

$$\frac{r^2}{\chi} \frac{d^2\chi}{dr^2} + \frac{r}{\chi} \frac{d\chi}{dr} = \frac{-1}{Y} \frac{d^2Y}{d\phi^2}.$$

We see that a function of r alone (the left side) is equal to a function of ϕ alone (the right side). Since we can vary r without changing ϕ , i.e., without changing the right side of the above equation, it must be that the left side of the above equation does not vary with r either. This means the left side of the above equation is not really a function of r but a constant. As a consequence, the right side is the same constant. Thus, for some $-\lambda$ we can write

$$\frac{-1}{Y} \frac{d^2Y}{d\phi^2} = -\lambda = \frac{r^2}{\chi} \frac{d^2\chi}{dr^2} + \frac{r}{\chi} \frac{d\chi}{dr}. \quad (1.8)$$

We solve $Y'' = \lambda Y$ to get the linearly independent solutions

$$Y(\phi) = \begin{cases} e^{\sqrt{\lambda}\phi}, e^{-\sqrt{\lambda}\phi} & \text{if } \lambda > 0, \\ 1, \phi & \text{if } \lambda = 0, \\ \sin(\sqrt{|\lambda|}\phi), \cos(\sqrt{|\lambda|}\phi) & \text{if } \lambda < 0, \end{cases}$$

but we must reject some of these solutions. Since (r_0, ϕ_0) represents the same point as $(r_0, \phi_0 + 2\pi k)$ for any $k \in \mathbb{Z}$, we require $Y(\phi)$ to have period 2π . Thus, we can only accept the linearly independent periodic solutions $1, \sin(\sqrt{|\lambda|}\phi)$, and $\cos(\sqrt{|\lambda|}\phi)$, where $\sqrt{|\lambda|}$ must be an integer. Then, let us replace λ with $-m^2$ and write our linearly independent solutions to $Y'' = -m^2 Y$ as

$$Y_{1,n} = \cos(n\phi), \quad Y_{2,m} = \sin(m\phi), \quad (1.9)$$

where³ $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Notice from (1.6) that $\partial^2/\partial\phi^2$ is the angular part of the Laplace operator in two dimensions and that the solutions given in (1.9) are eigenfunctions of the $\partial^2/\partial\phi^2$ operator. We will see in Chapter 4 that the functions in (1.9) are actually spherical harmonics; however, since we have not yet given a definition of a spherical harmonic, for now we will just refer to these as functions Y . The reader should keep in mind that characteristics of the Y 's we comment on here will generalize when we move to \mathbb{R}^p .

³We use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

We can also solve easily for the functions $\chi(r)$ that satisfy (1.8), but this does not concern us here. Let us instead notice a few properties of the functions Y . Let us consider these to be functions $r^m \sin(m\phi)$, $r^n \cos(n\phi)$ on \mathbb{R}^2 that have been restricted to the unit circle, where $r = 1$, and let us analyze the extended functions on \mathbb{R}^2 . We will first rewrite them using *Euler's formula*, $e^{i\phi} = \cos \phi + i \sin \phi$, which implies

$$\begin{aligned} (x + iy)^n &= \left(r e^{i\phi} \right)^n = r^n e^{in\phi} = r^n [\cos(n\phi) + i \sin(n\phi)], \\ (x - iy)^n &= \left(r e^{-i\phi} \right)^n = r^n e^{-in\phi} = r^n [\cos(n\phi) - i \sin(n\phi)], \end{aligned}$$

so that

$$\begin{aligned} r^n \cos(n\phi) &= \frac{1}{2} [(x + iy)^n + (x - iy)^n] \stackrel{\text{def}}{=} H_{1,n}(x, y), \\ r^n \sin(n\phi) &= \frac{1}{2i} [(x + iy)^n - (x - iy)^n] \stackrel{\text{def}}{=} H_{2,n}(x, y). \end{aligned}$$

We notice that the Y 's can be written as polynomials restricted to the unit circle, where $r = 1$. Furthermore, observe that

$$H_{1,n}(tx, ty) = t^n H_{1,n}(x, y), \quad \text{and} \quad H_{2,n}(tx, ty) = t^n H_{2,n}(x, y);$$

we call polynomials with this property *homogeneous of degree n* . Moreover, the reader can also check that these polynomials satisfy the Laplace equation (1.3), by either using (1.1) or the Laplace operator in polar coordinates (1.6) for the computation, i.e.,

$$\Delta_2 H_{1,n} = 0, \quad \text{and} \quad \Delta_2 H_{2,n} = 0.$$

Let us also notice that the Y 's of different degree are *orthogonal over the unit circle*, which means

$$\begin{aligned} \int_0^{2\pi} \sin(n\phi) \sin(m\phi) d\phi &= 0, \quad \text{if } n \neq m, \\ \int_0^{2\pi} \cos(n\phi) \cos(m\phi) d\phi &= 0, \quad \text{if } n \neq m, \\ \int_0^{2\pi} \sin(n\phi) \cos(m\phi) d\phi &= 0, \quad \text{if } n \neq m, \end{aligned}$$

as we can easily compute by taking advantage of Euler's formula. For instance, we can calculate

$$\int_0^{2\pi} \sin(n\phi) \sin(m\phi) d\phi = \int_0^{2\pi} \frac{e^{in\phi} - e^{-in\phi}}{2i} \cdot \frac{e^{im\phi} - e^{-im\phi}}{2i} d\phi$$

which becomes

$$\int_0^{2\pi} \left(\frac{e^{i(m+n)\phi} - e^{-i(m+n)\phi}}{2} - \frac{e^{i(m-n)\phi} + e^{-i(m-n)\phi}}{2} \right) d\phi$$

or

$$\int_0^{2\pi} \left(\sin[(m+n)\phi] - \cos[(m-n)\phi] \right) d\phi = 0,$$

for $n \neq m$. The reader can check the rest in a similar fashion.

Finally, by recalling the theorems of Fourier analysis⁴, we know that any “reasonable” function defined on the unit circle can be expanded in a *Fourier series*. That is, given a function $f : [0, 2\pi) \rightarrow \mathbb{R}$ satisfying certain conditions (that do not concern us in this introduction), we can write

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin(m\phi) + \sum_{n=0}^{\infty} b_n \cos(n\phi)$$

for some constants a_m, b_n . We say that the Y ’s make up a *complete set* of functions over the unit circle, since we can expand any nice function $f(\phi)$ defined on $[0, 2\pi)$ in terms of them.

Before we move on, we will show that (1.4) can be approached using a method almost identical to the method we used above. Let us assume that the solution ψ to (1.4) can be written as $\psi(r, \phi) = \chi(r)Y(\phi)$. Then, using polar coordinates, the equation becomes

$$Y \left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - \frac{\hbar^2}{2mr} \frac{d}{dr} + V(r) \right) \chi + \frac{\chi}{r^2} \left(-\frac{\hbar^2}{2m} \frac{d^2}{d\phi^2} \right) Y = E\chi Y.$$

Multiplying by $r^2/\chi Y$ and rearranging,

$$\frac{1}{\chi} \left(-\frac{r^2 \hbar^2}{2m} \frac{d^2}{dr^2} - \frac{r \hbar^2}{2m} \frac{d}{dr} + r^2 V(r) \right) \chi - Er^2 = \frac{1}{Y} \left(\frac{\hbar^2}{2m} \frac{d^2}{d\phi^2} \right) Y.$$

Once again, we see that a function of r alone is equal to a function of ϕ alone, and we conclude that both sides of the above equation must be equal to the same constant. Using the same reasoning as before, we write this constant as $-\ell^2 \hbar^2 / 2m$, where ℓ is an integer. If we carry out the calculation, we will see that the functions $Y(\phi)$ are the same ones we found previously in this subsection. We will also find the radial equation

$$\frac{1}{\chi} \left(-\frac{r^2 \hbar^2}{2m} \frac{d^2}{dr^2} - \frac{r \hbar^2}{2m} \frac{d}{dr} + r^2 V(r) \right) \chi - Er^2 = -\frac{\ell^2 \hbar^2}{2m},$$

⁴Doing so will not be necessary to understand the material we present here, but the reader unfamiliar with Fourier analysis may choose to consult [3].

which we can rewrite as

$$\left\{ -\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} \right) + \left[V(r) + \frac{\ell^2 \hbar^2}{2mr^2} \right] \right\} \chi = E\chi.$$

It is interesting to note that this equation resembles (1.4). If we think of r as our single independent variable and χ as our wave function, we have an effective potential energy

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2 \hbar^2}{2mr^2},$$

where we call the second term the *centrifugal* term. Thus χ represents a fictitious particle that feels an effective force

$$\vec{F}_{\text{eff}} = -\nabla V_{\text{eff}} = -\frac{dV_{\text{eff}}}{dr} \hat{r}.$$

We see that the centrifugal term contributes a force

$$\vec{F}_{\text{centrifugal}} = \frac{\ell^2 \hbar^2}{mr^3} \hat{r},$$

pushing the particle away from the center of symmetry.

Three-Dimensional Case

Here, we will follow a procedure almost identical to that of the last subsection, but we will not take the discussion as far in three dimensions. The Y 's we will find in three dimensions are more widely used than those in any other number of dimensions; however, a thorough development of the functions in \mathbb{R}^3 could take many pages, and this would distract us from our goal of moving to p dimensions. Furthermore, the results that we would find in three dimensions are only special cases of more general theorems we will develop later in the discussion. If the reader is interested in studying the usual spherical harmonics of \mathbb{R}^3 in depth, there are a multitude of sources we can recommend, including [1], [2], [3] and [13].

Let us rewrite the Laplace operator in spherical coordinates, which are given by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right), \quad \phi = \tan^{-1} \left(\frac{y}{x} \right). \quad (1.10)$$

Alternatively,

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Using the chain rule, we find

$$\Delta_3 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]. \quad (1.11)$$

As in the two-dimensional case, it is probably easiest to verify this formula by starting with (1.11) and producing (1.2). The reader should check this result for practice, proceeding exactly as we did beginning in (1.7).

Inserting (1.11) into (1.3) gives

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \right] = 0.$$

Searching for solutions Φ in the form $\Phi(r, \theta, \phi) = \chi(r)Y(\theta, \phi)$, where $\chi(r)$ is a function of r alone and $Y(\theta, \phi)$ is a function of only θ and ϕ , this becomes

$$Y \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\chi}{dr} \right) \chi + \frac{\chi}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y = 0.$$

Multiplying by $r^2/\chi Y$ and rearranging,

$$\frac{1}{\chi} \frac{d}{dr} \left(r^2 \frac{d\chi}{dr} \right) \chi = \frac{-1}{Y} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y.$$

Since we have found that a function of r alone is equal to a function of only θ and ϕ , we use the same reasoning as in the previous subsection to conclude that both sides of the above equation must be equal to the same constant, call it $-\lambda$. This implies that

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y = \lambda Y. \quad (1.12)$$

At this point, we will stop. It turns out that the Y 's which satisfy this equation are actually spherical harmonics. Comparing (1.11) and (1.12), we see that the Y 's are eigenfunctions of the angular part of the Laplace operator, just as in two dimensions. If we studied the functions Y further we would also find that, analogously to what we noticed in \mathbb{R}^2 , the Y 's form a complete set over the unit sphere, each Y is a homogeneous polynomial restricted to the unit sphere, and these polynomials satisfy the Laplace equation. However, to develop these results and the many others that exist would require us to study Legendre's equation, Legendre polynomials, and associated Legendre functions, and we will choose to leave such an in-depth analysis to the general case of p dimensions.

We will now move on to see how these functions Y are related to angular momentum in quantum mechanics.

1.2 Quantum Mechanical Angular Momentum

We have seen that spherical harmonics in two and three dimensions relate to the one-dimensional sphere (circle) and the two-dimensional sphere (surface of a regular ball) respectively. In quantum mechanics, rotations of a system are generated by the angular momentum operator. Spherical symmetry means invariance under all such rotations. Therefore, a relation between the theory of spherical harmonics and the theory of angular momentum is not only expected but is a natural and fundamental result.

Recall that in classical mechanics, the angular momentum of a particle is defined by the cross product

$$\vec{L} = \vec{r} \times \vec{p},$$

where \vec{p} is its linear momentum. To find the quantum mechanical angular momentum operator, we make the substitution $p_i \mapsto -i\hbar \partial/\partial x_i$ where $x_1 = x$, $x_2 = y$, and $x_3 = z$. Thus, we see that⁵

$$\hat{\vec{L}} = -i\hbar \vec{r} \times \nabla,$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Two-Dimensional Case

In the plane, the angular momentum operator has only one component, given by

$$\hat{L} = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).$$

Using the polar coordinates defined in (1.5) and the chain rule, we can rewrite this as

$$\hat{L} = -i\hbar \frac{\partial}{\partial \phi},$$

as the reader should have no problem checking using the same strategy s/he used to verify (1.6) and (1.11). Then

$$\hat{L}^2 = -\hbar^2 \frac{\partial^2}{\partial \phi^2},$$

and we check using (1.9) that the functions $Y(\phi)$ are eigenfunctions of the \hat{L}^2 operator. In particular,

$$\hat{L}^2 Y_{m,j}(\phi) = -\hbar^2 \frac{\partial^2}{\partial \phi^2} Y_{m,j} = \hbar^2 m^2 Y_{m,j},$$

⁵The hat above the angular momentum indicates that it is an operator (not a unit vector) in this section.

and we see that the function $Y_{m,j}$ is associated with the eigenvalue $\hbar^2 m^2$.

In quantum mechanics, operators such as \hat{L} represent dynamical variables. If an operator \hat{O} has eigenfunctions ψ_k with corresponding eigenvalues λ_k , then a particle in state ψ_k will be observed to have a value of λ_k for the dynamical variable \hat{O} . Therefore, we see that a particle in the state $Y_{m,j}$ will be observed to have a value of $\hbar^2 m^2$ for its angular momentum squared. We say the function $Y_{m,j}$ carries angular momentum $\hbar m$.

Three-Dimensional Case

Things work similarly in three dimensions, where the angular momentum operator has the components

$$\begin{aligned}\hat{L}_x &= -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \\ \hat{L}_y &= -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \\ \hat{L}_z &= -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).\end{aligned}$$

Using the spherical coordinates defined in (1.10) and the chain rule, we can rewrite these as

$$\begin{aligned}\hat{L}_x &= i\hbar \left(\sin \theta \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi} \right), \\ \hat{L}_y &= -i\hbar \left(\cos \theta \frac{\partial}{\partial \theta} + \sin \phi \cot \theta \frac{\partial}{\partial \phi} \right), \\ \hat{L}_z &= -i\hbar \frac{\partial}{\partial \phi};\end{aligned}$$

the reader should check these formulas. These equations allow us to compute $\hat{L}^2 = \hat{L} \cdot \hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. Carrying out the multiplication,

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right], \quad (1.13)$$

and we see that by (1.12), the functions Y are eigenfunctions of the \hat{L}^2 operator.

We claimed in the last section that the functions $Y(\theta, \phi)$ were homogeneous polynomials with restricted domain, so let us write $Y_\ell(\theta, \phi)$ where ℓ denotes the degree of homogeneity. In Section 4.2, we will see an easy way to compute the eigenvalue of \hat{L}^2 associated with Y_ℓ , and it will turn out to be $\hbar^2 \ell(\ell + 1)$.

So we claim that in three dimensions, the function $Y_\ell(\theta, \phi)$ carries an angular momentum of $\hbar\sqrt{\ell(\ell+1)}$.

In Chapter 4, we will give rigorous foundations to the seemingly coincidental facts we have discovered in this chapter about the functions Y that arose as solutions to certain differential equations. But first, we will devote a chapter to gaining some practice and intuition working in \mathbb{R}^p .

Chapter 2

Working in p Dimensions

In this chapter, we spend some time developing our skills in performing calculations in \mathbb{R}^p and exercising our abilities in visualizing a p -dimensional space for an arbitrary natural number p . We will use the majority of the results we obtain here in the development of our main subject, but some topics we discuss just out of pure interest or to improve our intuition.

First, let us generalize the definition of the Laplace operator to \mathbb{R}^p , where a point¹ x is given by the ordered pair (x_1, x_2, \dots, x_p) .

Definition The *Laplace operator* in \mathbb{R}^p is given by

$$\Delta_p \stackrel{\text{def}}{=} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \quad (2.1)$$

The *del operator* in \mathbb{R}^p is the vector operator

$$\nabla_p \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_p} \right).$$

2.1 Rotations in \mathbb{R}^p

Let us quickly consider orthogonal rotations of the coordinate axes in \mathbb{R}^p . Such rotations leave the lengths of vectors unchanged. Indeed, the length of a vector is a geometric quantity; rotating the coordinate system we use to describe the vector leaves its length invariant. In fact, in a more abstract setting, we could define a rotation to be any transformation of coordinates that leaves the lengths of vectors unchanged.

In what follows, we let x denote a column vector² $(x_1, x_2, \dots, x_p)^t$ in \mathbb{R}^p and use $\langle \cdot, \cdot \rangle$ to represent the dot product of two vectors. The fact that a

¹We will not place vector arrows above points x in \mathbb{R}^p .

²The superscript t denotes the operation of matrix transposition.

rotation matrix R leaves the length of x invariant means $\langle Rx, Rx \rangle = \langle x, x \rangle$. Moreover, since the dot product between any two vectors x, y can be written as

$$\langle x, y \rangle = \frac{1}{2} (\langle x + y, x + y \rangle - \langle x, x \rangle - \langle y, y \rangle),$$

it follows that coordinate rotations leave all dot products invariant.

Notice further that we can write dot products such as $\langle x, y \rangle$ as matrix products $y^t x$. In this notation, the requirement that $\langle Rx, Ry \rangle = \langle x, y \rangle$ translates into the necessity of $(Ry)^t (Rx) = y^t x$, or $y^t R^t R x = y^t x$. Since this equation must hold for all $x, y \in \mathbb{R}^p$, we can conclude that any rotation matrix R must satisfy $R^t R = I$, where I is the identity matrix.

Now we can verify our claim in the first few sentences of this booklet that the Laplace operator Δ_p remains unchanged after being subjected to a rotation of coordinates.

Proposition 2.1 *The Laplace operator Δ_p is invariant under coordinate rotations. That is, if R is a rotation matrix and $x' = Rx$, then $\Delta'_p = \Delta_p$, i.e.*

$$\sum_{j=1}^p \left(\frac{\partial}{\partial x'_j} \right)^2 = \sum_{j=1}^p \left(\frac{\partial}{\partial x_j} \right)^2.$$

Proof This can be proved very easily by noticing that $\Delta_p = \nabla_p \cdot \nabla_p$ is a dot product of vector operators. Since all dot products are unchanged by coordinate rotations, we can conclude that Δ_p is not affected by any rotation R .

In case the reader is not satisfied with this quick justification, let us compute Δ'_p , the Laplace operator after application of a rotation of coordinates R . Since R is a rotation matrix, it is orthogonal, i.e. $RR^t = I$. Then, using the chain rule,

$$\begin{aligned} \Delta_p &= \sum_{j=1}^p \left(\frac{\partial}{\partial x_j} \right)^2 = \sum_{j=1}^p \left[\left(\sum_{k=1}^p \frac{\partial x'_k}{\partial x_j} \frac{\partial}{\partial x'_k} \right) \left(\sum_{\ell=1}^p \frac{\partial x'_\ell}{\partial x_j} \frac{\partial}{\partial x'_\ell} \right) \right] \\ &= \sum_{j=1}^p \left[\left(\sum_{k=1}^p R_{kj} \frac{\partial}{\partial x'_k} \right) \left(\sum_{\ell=1}^p R_{\ell j} \frac{\partial}{\partial x'_\ell} \right) \right], \end{aligned}$$

so

$$\Delta_p = \sum_{k,\ell=1}^p \frac{\partial}{\partial x'_k} \frac{\partial}{\partial x'_\ell} \left(\sum_{j=1}^p R_{kj} R_{\ell j} \right) = \sum_{k=1}^p \left(\frac{\partial}{\partial x'_k} \right)^2 = \Delta'_p.$$

This proves the proposition. ■

2.2 Spherical Coordinates in p Dimensions

Now, in order to develop some experience and intuition working in higher-dimensional spaces, we will develop the spherical coordinate system for \mathbb{R}^p in considerable detail. In particular, we will use an inductive technique to come up with the expression of the spherical coordinates in p dimensions in terms of the corresponding Cartesian coordinates.

We will let our space have axes denoted x_1, x_2, \dots . First, in two dimensions, spherical coordinates are just the polar coordinates given in (1.5),

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \in [0, \infty), \\ \phi &= \tan^{-1}(x_2/x_1) \in [0, 2\pi), \end{aligned}$$

where r is the distance from the origin and ϕ is the azimuthal angle³ in the plane that measures the rotation around the origin. The inverse transformation is

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi.$$

When we move to three dimensions we add an axis, naming it x_3 , perpendicular to the plane. Now, the polar coordinates above can only define a location in the plane; thus, they only tell us on which vertical line (i.e., line parallel to the x_3 -axis) we lie, as we can see in Figure 2.1 with $p = 3$. To pinpoint our location on this line, we introduce a new angle θ_1 . When we also redefine r to be the three-dimensional distance from the origin, we have the spherical coordinates given in (1.10),

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2 + x_3^2} \in [0, \infty), \\ \phi &= \tan^{-1}(x_2/x_1) \in [0, 2\pi), \\ \theta_1 &= \tan^{-1}\left(\sqrt{x_1^2 + x_2^2}/x_3\right) \in [0, \pi]. \end{aligned}$$

Now let us imagine moving to four dimensions by adding an axis — name it the x_4 -axis — perpendicular to the three-dimensional space just discussed. The 3D spherical coordinates given above can only define a location in \mathbb{R}^3 , so they only tell us on which “vertical” line (i.e., line parallel to the x_4 -axis) we lie, as in Figure 2.1 with $p = 4$. We thus introduce a new angle θ_2 to determine the location on this line. We redefine r to be the four-dimensional distance from

³We keep the definition of ϕ sloppy throughout this section. A more precise formula would use a two-argument \tan^{-1} function that produces angles on the entire unit circle.

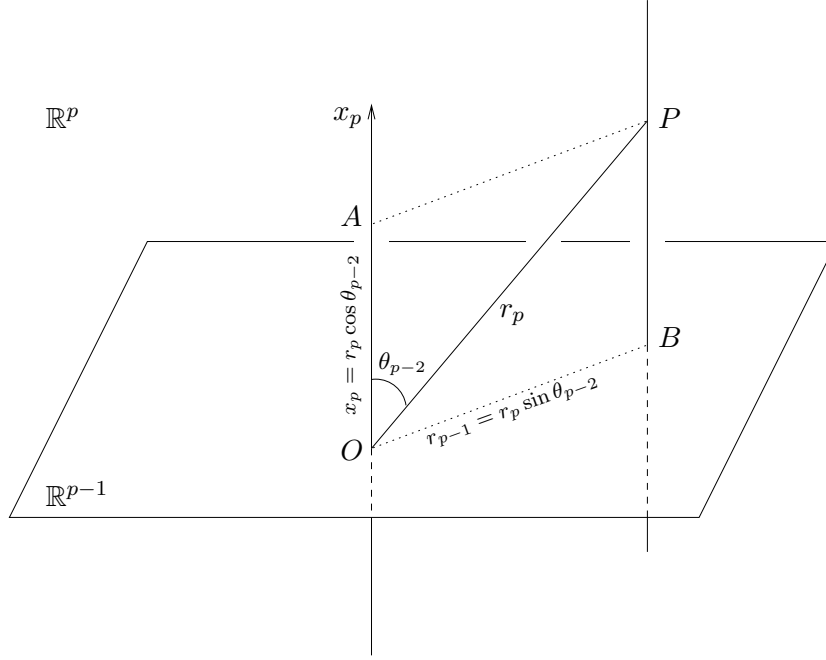


Figure 2.1: In going from \mathbb{R}^{p-1} to \mathbb{R}^p , we visualize \mathbb{R}^{p-1} as a plane and add a new perpendicular direction. We introduce a new angular coordinate θ_{p-2} to determine the location in the new direction.

the origin, and this completes the construction of the 4D spherical coordinates,

$$\begin{aligned}
 r &= \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} && \in [0, \infty), \\
 \phi &= \tan^{-1}(x_2/x_1) && \in [0, 2\pi), \\
 \theta_1 &= \tan^{-1}\left(\sqrt{x_1^2 + x_2^2}/x_3\right) && \in [0, \pi], \\
 \theta_2 &= \tan^{-1}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}/x_4\right) && \in [0, \pi],
 \end{aligned}$$

where it should be clear from the figure that the new angle θ_2 only ranges from 0 to π .

In going from three to four dimensions, we mimicked the way in which we transitioned from two to three dimensions. We follow the same procedure each

time we move up in dimension. For arbitrary p , this yields

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}, \\ \phi &= \tan^{-1}(x_2/x_1), \\ \theta_1 &= \tan^{-1}\left(\sqrt{x_1^2 + x_2^2}/x_3\right), \\ &\vdots \\ \theta_{p-2} &= \tan^{-1}\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_{p-1}^2}/x_p\right), \end{aligned}$$

where the ranges on the coordinates are as expected from the previous cases.

We have thus defined the spherical coordinates in p -dimensions in terms of the corresponding Cartesian coordinates. To write down the inverse relations we use projections, with Figure 2.1 as an aid. As before, we will derive these relations in detail for a few instructive cases before writing down the most general expressions.

We have already written down x_1, x_2 in terms of r, ϕ so let's use \mathbb{R}^3 as our first example. We imagine \mathbb{R}^3 as the direct sum of a two-dimensional plane \mathbb{R}^2 with the real line \mathbb{R} . Then, given the point P in \mathbb{R}^3 with spherical coordinates (r_3, ϕ, θ_1) , the vector \overrightarrow{OP} can be written as a sum $\overrightarrow{OA} + \overrightarrow{OB}$, where \overrightarrow{OA} lies along the x_3 -axis and has magnitude $r_3 \cos \theta_1$ and \overrightarrow{OB} lies in the plane and has magnitude $r_2 = r_3 \sin \theta_1$. The point B thus has spherical coordinates (r_2, ϕ) in the plane, implying

$$x_1 = r_2 \cos \phi \quad \text{and} \quad x_2 = r_2 \sin \phi,$$

so

$$x_1 = r_3 \sin \theta_1 \cos \phi, \tag{2.2}$$

$$x_2 = r_3 \sin \theta_1 \sin \phi, \tag{2.3}$$

$$x_3 = r_3 \cos \theta_1. \tag{2.4}$$

Let us examine the situation in \mathbb{R}^4 using the same technique. Given the point P with radial distance r_4 from the origin, we decompose \overrightarrow{OP} into two vectors $\overrightarrow{OA} + \overrightarrow{OB}$, where \overrightarrow{OA} lies along the x_4 -axis and has magnitude $r_4 \cos \theta_2$ and \overrightarrow{OB} lies in the “plane” and has magnitude $r_3 = r_4 \sin \theta_2$. The point B thus has spherical coordinates (r_3, ϕ, θ_1) in the three-dimensional space, so x_1, x_2, x_3 are as in (2.2)–(2.4). Therefore

$$x_1 = r_4 \sin \theta_2 \sin \theta_1 \cos \phi,$$

$$x_2 = r_4 \sin \theta_2 \sin \theta_1 \sin \phi,$$

$$x_3 = r_4 \sin \theta_2 \cos \theta_1,$$

$$x_4 = r_4 \cos \theta_2.$$

If the reader has understood the previous constructions, the expressions in p dimensions should be evident. Given the radius r_p in \mathbb{R}^p , we project the position vector onto \mathbb{R}^{p-1} , obtaining the radius in the subspace \mathbb{R}^{p-1} given by $r_{p-1} = r_p \sin \theta_{p-2}$. In this way we can perform a series of projections to get down to a space in which we already know the relations. This procedure leads to (dropping the subscript on r_p)

$$\begin{aligned} x_1 &= r \sin \theta_{p-2} \sin \theta_{p-3} \cdots \sin \theta_3 \sin \theta_2 \sin \theta_1 \cos \phi, \\ x_2 &= r \sin \theta_{p-2} \sin \theta_{p-3} \cdots \sin \theta_3 \sin \theta_2 \sin \theta_1 \sin \phi, \\ x_3 &= r \sin \theta_{p-2} \sin \theta_{p-3} \cdots \sin \theta_3 \sin \theta_2 \cos \theta_1, \\ x_4 &= r \sin \theta_{p-2} \sin \theta_{p-3} \cdots \sin \theta_3 \cos \theta_2, \\ &\vdots \\ x_{p-1} &= r \sin \theta_{p-2} \cos \theta_{p-3}, \\ x_p &= r \cos \theta_{p-2}. \end{aligned}$$

With this chapter as an exception, we will rarely refer explicitly to the angles $\phi, \theta_1, \dots, \theta_{p-2}$. However, we will frequently use $r = \sqrt{x_1^2 + \cdots + x_p^2}$.

2.3 The Sphere in Higher Dimensions

We will give definitions of the sphere and the ball in an arbitrary number of dimensions that are analogous to the definitions of the familiar sphere and ball we visualize embedded in \mathbb{R}^3 .

Definition The $(p-1)$ -sphere of radius δ centered at x_0 is the set

$$S_\delta^{p-1}(x_0) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^p : |x - x_0| = \delta\}.$$

The *unit $(p-1)$ -sphere*⁴ is the set $S^{p-1} \stackrel{\text{def}}{=} S_1^{p-1}(0)$.

Definition The *open p -ball of radius δ centered at x_0* is the set

$$B_\delta^p(x_0) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^p : |x - x_0| < \delta\}.$$

The *open unit p -ball* is the set $B^p \stackrel{\text{def}}{=} B_1^p(0)$. The *closed p -ball* is the set $\bar{B}^p \stackrel{\text{def}}{=} B^p \cup S^{p-1}$.

Notice we call the sphere that we picture embedded in \mathbb{R}^p the $(p-1)$ -sphere because it is $(p-1)$ -dimensional. It requires $p-1$ angles to define one's location on the sphere, as we saw in Section 2.2. However, it requires p

⁴Frequently, we will drop the “unit,” though it is still implied.

coordinates to locate a point in the ball because we must also specify r ; this justifies the notation B^p . Notice also that the $(p-1)$ -sphere is the boundary of the p -ball when we think of these as subsets of \mathbb{R}^p ; we write $S^{p-1} = \partial B^p$.

Let us compute the surface area of the $(p-1)$ -sphere. Towards this goal, we recall that the gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt,$$

for any $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. Then

Lemma 2.2 *For all $p \in \mathbb{C}$ such that $\operatorname{Re}(p) > 0$, we have*

$$\int_0^{+\infty} e^{-r^2} r^{p-1} dr = \frac{1}{2} \Gamma\left(\frac{p}{2}\right).$$

Proof Using the substitution $u = r^2$,

$$\int_0^{+\infty} e^{-r^2} r^{p-1} dr = \int_0^{\infty} e^{-u} u^{\frac{p-1}{2}} \frac{dr}{2\sqrt{u}} = \frac{1}{2} \int_0^{+\infty} e^{-u} u^{\frac{p}{2}-1} du = \frac{1}{2} \Gamma\left(\frac{p}{2}\right),$$

as sought. ■

Proposition 2.3 *If Ω_{p-1} denotes the solid angle in \mathbb{R}^p (equivalent numerically to the surface area of S^{p-1}), then*

$$\Omega_{p-1} = \frac{2\pi^{p/2}}{\Gamma(p/2)}.$$

Before proving this we will digress slightly and discuss how the surface area of the $(p-1)$ -sphere relates to the volume of the p -ball. First, consider the p -ball of radius r centered at the origin. Notice that the radius r completely determines such a ball. If we were to determine the volume V_p of this p -ball, we would obtain $V_p = cr^p$ where c is some constant. Here, we have determined that $V_p \propto r^p$ because V_p must have dimensions of $[\text{length}]^p$ and the only variable characterizing the p -ball is r (which has dimensions of length). Now if we differentiate V_p with respect to r , we get

$$\frac{dV_p}{dr} = (p-1) cr^{p-1}.$$

Thus, if the radius of a p -ball changes by an infinitesimal amount δr , its volume will change by some infinitesimal amount δV_p , and

$$\delta V_p = (p-1) c r^{p-1} \delta r. \quad (2.5)$$

But in this case, the small change in volume δV should equal the surface area $A_{p-1}(r)$ of the p -ball multiplied by the small change in radius δr ,

$$\delta V_p = A_{p-1}(r) \delta r. \quad (2.6)$$

From (2.5) and (2.6) we see that

$$A_{p-1}(r) = (p-1) c r^{p-1},$$

and if we let $\Omega_{p-1} = (p-1) c$ denote the numerical value of the surface area when $r = 1$ (as in the statement of Proposition 2.3), we get⁵

$$A_{p-1}(r) = \Omega_{p-1} r^{p-1}. \quad (2.7)$$

We see that if we want to carry out an integral over \mathbb{R}^p when the integrand depends only on r , we can use the differential volume element

$$dV_p = A_{p-1}(r) dr = r^{p-1} \Omega_{p-1} dr.$$

Now we are ready to prove Proposition 2.3.

Proof Consider the integral,

$$J = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_p e^{-(x_1^2 + x_2^2 + \cdots + x_p^2)}.$$

This is really

$$J = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^p = (\sqrt{\pi})^p.$$

Using spherical coordinates however, we can write

$$J = \int_{S^{p-1}} \int_0^{+\infty} dV_p e^{-r^2} = \Omega_{p-1} \int_0^{\infty} e^{-r^2} r^{p-1} dr,$$

⁵We should stress that, since a physicist assigns physical dimensions to each and every quantity, s/he would differentiate between the solid angle Ω_{p-1} and the surface area A_{p-1} of the $(p-1)$ -sphere. Although numerically the two quantities are equal for the unit sphere since $r = 1$, they are different quantities since Ω_{p-1} is dimensionless while A_{p-1} has dimensions of $[\text{length}]^{p-1}$.

since Ω_{p-1} is just a constant. Therefore

$$\Omega_{p-1} = \frac{\pi^{p/2}}{\int_0^\infty e^{-r^2} \cdot r^{p-1} dr}$$

and, with the help of Lemma 2.2 we arrive at the advertised result. \blacksquare

Remark As a check, we can use this formula to determine the surface area (or circumference) of the 1-sphere (or circle) as well as the surface area of the 2-sphere, which is the familiar sphere that we embed in \mathbb{R}^3 . As expected,

$$\Omega_1 = 2\pi, \quad \Omega_2 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = 4\pi,$$

using $\Gamma(3/2) = \sqrt{\pi}/2$. It is interesting to consider the 0-sphere, i.e., the sphere that we visualize embedded in \mathbb{R} . S^0 consists of all points on the real line that are unit distance from the origin, so $S^0 = \{-1, 1\}$. Using Lemma 2.3, we find $\Omega_0 = 2$, since $\Gamma(1/2) = \sqrt{\pi}$. That is, the surface area of just two points on the real line is finite! This is consistent with standard calculus, though. On the real line, the radial distance is the absolute value of x . Hence, the concept of a function $f(r)$ possessing spherical symmetry coincides with the concept of an even function, $f(x) = f(-x)$. Then

$$\int_{-\infty}^{+\infty} f(x) dx = 2 \int_0^\infty f(x) dx = \int_0^\infty f(r) \Omega_0 r^0 dr.$$

2.4 Arc Length in Spherical Coordinates

Now, let us compute the formula for the infinitesimal arc length δs associated with a small displacement along a curve in \mathbb{R}^p . In this section, we use infinitesimals in the physicist's way; that is, δs and the other small lengths involved represent tiny displacements that are small enough so that the errors involved in our approximations (such as assuming a small portion of a curve is almost straight) are as small as desired (say, less than some given $\epsilon > 0$). We only use the formulas we derive under this assumption in the limit as $\delta s \rightarrow 0$ and likewise with the other infinitesimal quantities, so our approximations introduce no error into future calculations.

Working in \mathbb{R}^p , let us use the spherical coordinates we developed in Section 2.2. Consider a point P with coordinates $(r, \phi, \theta_1, \dots, \theta_{p-2})$. Consider a small displacement vector $\delta \vec{s} = \overrightarrow{PP'}$ at the point P in an arbitrary direction. One component of this displacement is along the radial direction and has magnitude

δr . The other $p - 1$ components of $\delta \vec{s}$ are orthogonal to $\delta \vec{r}$ and thus⁶ are along the surface $S_r^{p-1}(0)$. Thus by the *Pythagorean Theorem*, we have so far $\delta s^2 = \delta r^2 + \dots$ where the rest to be filled in involves small changes in the angles $\theta_{p-2}, \dots, \theta_1, \phi$.

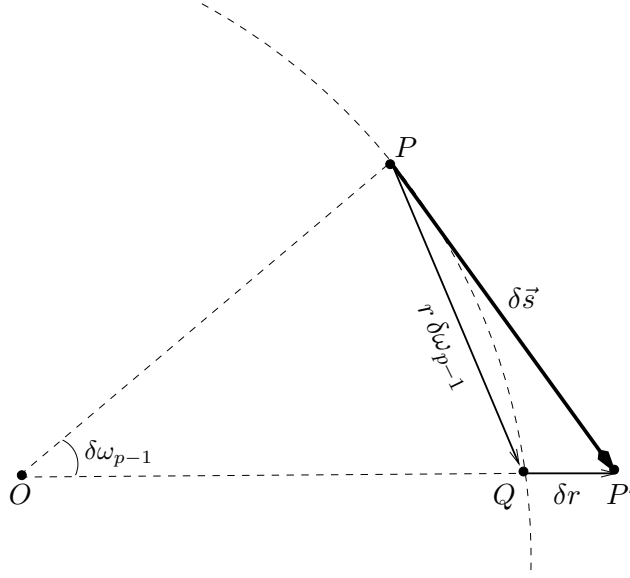


Figure 2.2: The plane containing the points O, P, P' . The displacement vector $\delta \vec{s}$ has a radial component of magnitude δr and a component along the $(p - 1)$ -sphere of magnitude $r \delta \omega_{p-1}$.

Let us consider the plane containing the vectors \overrightarrow{OP} and $\overrightarrow{OP'}$, where O is the origin (see Figure 2.2). We are familiar with the formula for an infinitesimal arc length in this two-dimensional plane, namely

$$\delta s^2 = \delta r^2 + r^2 \delta \omega_{p-1}^2, \quad (2.8)$$

where $\delta \omega_{p-1}$ is the small angle between the vectors \overrightarrow{OP} and $\overrightarrow{OP'}$. Note $\delta \omega_{p-1}$ is the angular displacement that occurs along the unit $(p - 1)$ -sphere.

Let us constrain the displacement $\delta \vec{s}$ to lie along a sphere by requiring $\delta r = 0$. If we compute δs in this special case, we have found $\delta \omega_{p-1} = \delta s / r$ which we can plug into (2.8) to obtain the general result.

We can determine $\delta \omega_{p-1}$ inductively, starting in the plane with $p = 2$. We

⁶The remaining components of $\delta \vec{s}$ lie in a surface where $\delta r = 0$ or $r = \text{const}$. This is a $(p - 1)$ -sphere.

know that on the unit circle $\delta s = r \delta \phi$, so

$$\delta \omega_1^2 = \frac{\delta s^2}{r^2} = \delta \phi^2.$$

In \mathbb{R}^3 , $\delta \vec{s}$ has a component $r \delta \theta_1$ along the direction of increasing θ_1 , so that $\delta s^2 = r^2 \delta \theta_1^2$ or $\delta \omega_2^2 = \delta \theta_1^2 + \dots$, and the remaining orthogonal components are parallel to the \mathbb{R}^2 plane. Projecting $\overrightarrow{PP'}$ onto \mathbb{R}^2 , we see that the infinitesimal displacement lies on a circle of radius $\sin \theta_1$. Using our result from \mathbb{R}^2 , we get

$$\delta \omega_2^2 = \delta \theta_1^2 + \sin^2 \theta_1 \delta \omega_1^2 = \delta \theta_1^2 + \sin^2 \theta_1 \delta \phi^2,$$

the familiar result in \mathbb{R}^3 .

We can move to \mathbb{R}^4 in an analogous fashion. An infinitesimal arc length $\delta \vec{s}$ on the 3-sphere has a component $r \delta \theta_2$ along the direction of increasing θ_2 , so that $\delta s^2 = r^2 \delta \theta_2^2$ or $\delta \omega_3^2 = \delta \theta_2^2 + \dots$, and the remaining orthogonal components are parallel to \mathbb{R}^3 which we picture as a plane. Projecting $\overrightarrow{PP'}$ onto \mathbb{R}^3 , we see that the infinitesimal displacement lies on a 2-sphere of radius $\sin \theta_2$, so

$$\delta \omega_3^2 = \delta \theta_2^2 + \sin^2 \theta_2 \delta \omega_2^2 = \delta \theta_2^2 + \sin^2 \theta_2 \delta \theta_1^2 + \sin^2 \theta_2 \sin^2 \theta_1 \delta \phi^2.$$

It is clear that this pattern continues, so that in \mathbb{R}^p ,

$$\delta \omega_{p-1} = \delta \theta_{p-2}^2 + \sin^2 \theta_{p-2} \delta \omega_{p-2}^2.$$

By filling in the form of $\delta \omega_{p-2}^2$ in this expression, we get

$$\delta \omega_{p-1} = \delta \theta_{p-2}^2 + \sin^2 \theta_{p-2} (\delta \theta_{p-3}^2 + \sin^2 \theta_{p-3} \delta \omega_{p-3}^2).$$

Continuing to expand this expression, we come up with

$$\begin{aligned} \delta \omega_{p-1} = & \delta \theta_{p-2}^2 + \sin^2 \theta_{p-2} \delta \theta_{p-3}^2 + \sin^2 \theta_{p-2} \sin^2 \theta_{p-3} \delta \theta_{p-4}^2 \\ & + \dots + \sin^2 \theta_{p-2} \sin^2 \theta_{p-3} \dots \sin^2 \theta_1 \delta \phi^2. \end{aligned}$$

Putting this into (2.8) completes our computation.

2.5 The Divergence Theorem in \mathbb{R}^p

Let us now return to the use of Cartesian coordinates to give an intuitive “derivation” of the divergence theorem in p dimensions, which we will use numerous times in Chapter 4. For brevity, our justification of this theorem will be rather physical. For a more mathematical treatment, the interested reader should consult a calculus book (for example, [12]) for the theorem in \mathbb{R}^3 and see if s/he can generalize the proof there to \mathbb{R}^p . For a rigorous proof of the divergence theorem in p dimensions, s/he may consult an analysis text (for example, [10]).

Theorem 2.4 Let \vec{F} be a continuously differentiable vector field defined in the neighborhood of some closed, bounded domain V in \mathbb{R}^p which has smooth boundary ∂V . Then⁷

$$\int_V \nabla \cdot \vec{F} d^p x = \oint_{\partial V} \vec{F} \cdot \hat{n} d\sigma, \quad (2.9)$$

where \hat{n} is the unit outward normal vector on ∂V and $d\sigma$ is the differential element of surface area on ∂V .

“Proof” We will interpret \vec{F} as the flux density of some p -dimensional fluid moving through the volume V . In unit time at a point x , the volume of fluid which flows past an arbitrarily oriented unit surface with unit normal vector \hat{n} is given by $\vec{F}(x) \cdot \hat{n}$.

Let us fill the interior of V with a “grid” of disjoint boxes, none intersecting the boundary ∂V — see Figure 2.3. If the boxes are comparable in size to V , they will not make a complete covering of the interior, since many regions of V near ∂V will remain uncovered; however, as the lengths of the box edges approach zero, the entire interior of V can be covered by these boxes. We

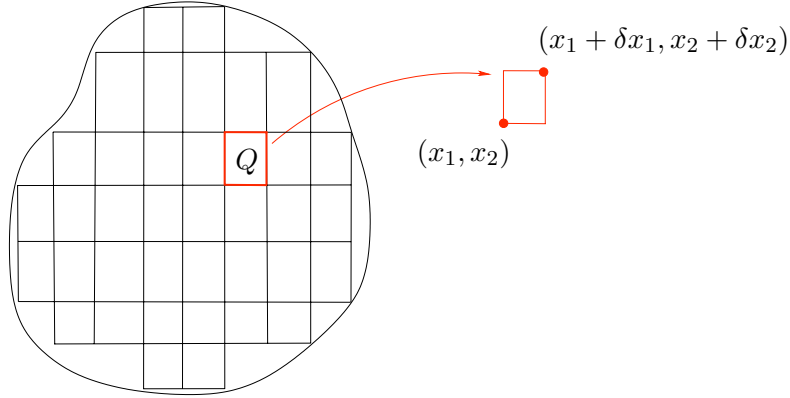


Figure 2.3: An example of a domain V in \mathbb{R}^2 . A similar picture would describe the situation in arbitrary \mathbb{R}^p . We consider one box Q with bottom-left corner at (x_1, x_2) and top-right corner at $(x_1 + \delta x_1, x_2 + \delta x_2)$.

will consider one small box Q in the interior of V , say with one corner at (x_1, x_2, \dots, x_p) and another at $(x_1 + \delta x_1, x_2 + \delta x_2, \dots, x_p + \delta x_p)$, where the

⁷Here and from now on, $d^p x = dx_1 dx_2 \cdots dx_p$.

δx_i are all positive. Consider the integral

$$J = \oint_{\partial Q} \vec{F} \cdot \hat{n} \, d\sigma.$$

The boundary ∂Q consists of $2p$ planes, two orthogonal to each coordinate axis. The two planes orthogonal to the x_1 axis have surface area equal to $\delta x_2 \delta x_3 \cdots \delta x_p$. Since \vec{F} is continuous, the integral of $\vec{F} \cdot \hat{n}$ over these surfaces can be written as⁸

$$\vec{F}(x_1 + \delta x_1, x_2, \dots, x_p) \cdot \hat{x}_1 \delta x_2 \delta x_3 \cdots \delta x_p,$$

and

$$\vec{F}(x_1, x_2, \dots, x_p) \cdot (-\hat{x}_1) \delta x_2 \delta x_3 \cdots \delta x_p,$$

since \vec{F} changes very little over this plane as $\delta x_1, \delta x_2, \dots, \delta x_p \rightarrow 0$. Now, there was nothing special about the x_1 -axis, so we see that for each x_i -axis, the integral J will include a term

$$\left[\vec{F}(x_1, \dots, x_i + \delta x_i, \dots, x_p) - \vec{F}(x_1, \dots, x_i, \dots, x_p) \right] \cdot \hat{x}_i \frac{\delta x_1 \cdots \delta x_p}{\delta x_i}.$$

This can be rewritten as

$$\left[\frac{\vec{F}(x_1, \dots, x_i + \delta x_i, \dots, x_p) - \vec{F}(x_1, \dots, x_i, \dots, x_p)}{\delta x_i} \right] \cdot \hat{x}_i \delta x_1 \cdots \delta x_p,$$

which becomes

$$\frac{\partial \vec{F}(x_1, \dots, x_p)}{\partial x_i} \cdot \hat{x}_i \delta x_1 \cdots \delta x_p \quad \text{or} \quad \frac{\partial F_i(x_1, \dots, x_p)}{\partial x_i} \delta x_1 \cdots \delta x_p$$

in the limit with which we are concerned. Putting all these terms together, we get

$$J = \sum_{i=1}^p \frac{\partial F_i(x_1, \dots, x_p)}{\partial x_i} \delta x_1 \cdots \delta x_p.$$

But this is just

$$\nabla_p \cdot \vec{F}(x_1, \dots, x_p) \delta x_1 \cdots \delta x_p = \int_Q \nabla \cdot \vec{F} \, d^p x$$

in our limit. We have thus shown that

$$\int_Q \nabla \cdot \vec{F} \, d^p x = \oint_{\partial Q} \vec{F} \cdot \hat{n} \, d\sigma$$

⁸Here, \hat{x}_1 is a unit vector in the direction of increasing x_1 .

for the special case of an infinitesimal box Q inside V . Let us now sum this result over all the boxes inside V ,

$$\sum_{Q_i} \int_{Q_i} \nabla \cdot \vec{F} d^p x = \sum_{Q_i} \oint_{\partial Q_i} \vec{F} \cdot \hat{n} d\sigma, \quad (2.10)$$

where we are concerned with the limit as the lengths of the box edges go to zero and the number of boxes in the covering approaches infinity. In this limit, the volume of the uncovered regions of V near ∂V approaches zero. Since \vec{F} is continuously differentiable, it follows that $\nabla \cdot \vec{F}$ is continuous and thus bounded over the closed and bounded region V . Thus, the integral of $\nabla \cdot \vec{F}$ over regions in V not covered by boxes approaches zero. The left side of (2.10) therefore becomes

$$\sum_{Q_i} \int_{Q_i} \nabla \cdot \vec{F} d^p x \longrightarrow \int_V \nabla \cdot \vec{F} d^p x \quad (2.11)$$

in this limit. Now, in the right side of (2.10), let us consider all the planes which bound the boxes Q_i that are included in the sum. Each “interior” plane appears twice in the sum, once as the “right” side of one box and a second time as the “left” side of another box. In each of these appearances, both \vec{F} and $d\sigma$ remain the same but the vector \hat{n} shows up with opposite sign. Thus, the integral over all the interior planes vanishes. The only terms that are not canceled in the integral on the right side of (2.10) make up the integral over the “exterior” planes, which we write as $\partial \bigcup_i Q_i$. That is,

$$\sum_{Q_i} \oint_{\partial Q_i} \vec{F} \cdot \hat{n} d\sigma = \oint_{\partial \bigcup_i Q_i} \vec{F} \cdot \hat{n} d\sigma,$$

so using (2.11),

$$\int_V \nabla \cdot \vec{F} d^p x = \oint_{\partial \bigcup_i Q_i} \vec{F} \cdot \hat{n} d\sigma. \quad (2.12)$$

Now we are very close to (2.9), but there is one difficulty due to the fact that the outward normal vector \hat{n} for $\partial \bigcup_i Q_i$ always points along one of the coordinate axes while \hat{n} can point in any direction for ∂V . To reconcile this difference, we realize that integrating $\vec{F} \cdot \hat{n}$ over a closed surface gives us the volume of fluid that has passed through this surface in unit time. The same volume of fluid must pass through both $\partial \bigcup_i Q_i$ and ∂V since $\partial \bigcup_i Q_i \rightarrow \partial V$. Therefore,

$$\oint_{\partial \bigcup_i Q_i} \vec{F} \cdot \hat{n} d\sigma = \oint_{\partial V} \vec{F} \cdot \hat{n} d\sigma.$$

Combining this with (2.12) completes the proof. ■

2.6 Δ_p in Spherical Coordinates

To compute Δ_p in spherical coordinates, we could use the chain rule. This would involve converting all the derivatives with respect to x_i in (2.1) to derivatives with respect to spherical coordinates as we did for Δ_2 and Δ_3 in Section 1.1. Such an undertaking would be quite messy, however, and adds little additional insight.

Another approach would be to use the general formula from differential geometry

$$\Delta = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu ,$$

where $g_{\mu\nu}$ is the metric tensor of the space of interest, $g = \det[g_{\mu\nu}]$ and $g^{\mu\nu}$ the inverse of the metric tensor. Since we have avoided this route thus far, we will not use it here either.

We can actually use an integration trick to determine the form of the p -dimensional Laplace operator Δ_p in spherical coordinates. This is the approach followed below.

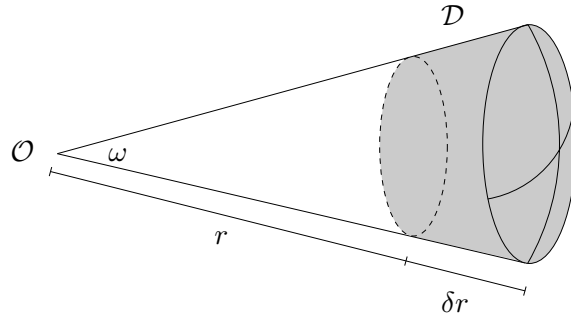


Figure 2.4: A region \mathcal{D} in \mathbb{R}^p .

Consider the p -dimensional cone-like region depicted in Figure 2.4. Let \mathcal{D} be the shaded region (a truncated cone), which has left and right boundaries given by portions of $S_r^{p-1}(0)$ and $S_{r+\delta r}^{p-1}(0)$ respectively. We are concerned with the limit where the quantity $\delta r \rightarrow 0$. This region spans a solid angle of ω , which means the surface area of the region is a fraction ω/Ω_{p-1} of the surface area of a complete sphere of the same radius. Using (2.7), this implies that the surface area of the left boundary is ωr^{p-1} while that of the right boundary is $\omega(r + \delta r)^{p-1}$.

Now let $f : \mathbb{R}^p \rightarrow \mathbb{R}$ be a twice continuously differentiable function, and

use the divergence theorem in p dimensions to find⁹

$$\int_{\mathcal{D}} \Delta_p f \, d^p x = \int_{\mathcal{D}} \nabla_p \cdot (\nabla_p f) \, d^p x = \int_{\partial \mathcal{D}} \nabla_p f \cdot \hat{n} \, d\sigma = \int_{\partial \mathcal{D}} \frac{\partial f}{\partial n} \, d\sigma,$$

where \hat{n} is the external unit normal vector to \mathcal{D} and $d\sigma$ is a differential element of surface area. By the *mean value theorem for integration*,

$$\int_{\mathcal{D}} \Delta_p f \, d^p x = \Delta_p f(x^*) \int_{\mathcal{D}} d^p x = \Delta_p f(x^*) \cdot \text{vol}(\mathcal{D}),$$

for some x^* in \mathcal{D} . But in the limit as the region of integration becomes infinitesimally small, it does not matter which x^* we use in \mathcal{D} since f is continuous, so

$$\int_{\mathcal{D}} \Delta_p f \, d^p x \rightarrow \Delta_p f \cdot \text{vol}(\mathcal{D}) \quad \text{as} \quad \text{vol}(\mathcal{D}) \rightarrow 0.$$

Thus

$$\Delta_p f = \lim_{\text{vol}(\mathcal{D}) \rightarrow 0} \left[\frac{1}{\text{vol}(\mathcal{D})} \int_{\partial \mathcal{D}} \left(\frac{\partial f}{\partial n} \right) d\sigma \right].$$

Note that in this limit, the volume of \mathcal{D} approaches $\omega r^{p-1} \delta r$. We can break up the integral in the numerator into one integral over the right bounding surface, one over the left bounding surface, and one over the lateral surface which we will denote by $\partial \mathcal{D}'$. Since in this limit the bounding surfaces are infinitesimally small, we can pull the integrands outside the integrals and rewrite the above equation as

$$\Delta_p f = \lim_{\substack{\delta r \rightarrow 0 \\ \omega \rightarrow 0}} \frac{\frac{\partial f}{\partial r}|_{r+\delta r} \omega (r+\delta r)^{p-1} - \frac{\partial f}{\partial r}|_r \omega r^{p-1}}{\omega r^{p-1} \delta r} + \lim_{\text{vol}(\mathcal{D}) \rightarrow 0} \left[\frac{\int_{\partial \mathcal{D}'} \left(\frac{\partial f}{\partial n} \right) d\sigma}{\text{vol}(\mathcal{D})} \right],$$

which becomes, using the binomial expansion,¹⁰

$$\lim_{\delta r \rightarrow 0} \frac{\frac{\partial f}{\partial r}|_{r+\delta r} [r^{p-1} + (p-1)r^{p-2}\delta r + \mathcal{O}(\delta r^2)] - \frac{\partial f}{\partial r}|_r r^{p-1}}{r^{p-1} \delta r} + \begin{array}{c} \text{contribution from} \\ \text{lateral surface,} \end{array}$$

or,

$$\Delta_p f = \lim_{\delta r \rightarrow 0} \left[\frac{\frac{\partial f}{\partial r}|_{r+\delta r} - \frac{\partial f}{\partial r}|_r}{\delta r} + \frac{p-1}{r} \frac{\partial f}{\partial r} \right] + \begin{array}{c} \text{contribution from} \\ \text{lateral surface.} \end{array} \quad (2.13)$$

⁹We introduce here our notation for a normal derivative: $\frac{\partial f}{\partial n} = \nabla_p f \cdot \hat{n}$.

¹⁰Here, $\mathcal{O}(\delta r^2)$ denotes terms which are such that $\frac{\text{terms}}{\delta r^2}$ remains finite as $\delta r \rightarrow 0$. In particular, $\frac{\text{terms}}{\delta r} \rightarrow 0$ as $\delta r \rightarrow 0$.

Notice that the contribution-from-the-lateral-surface term

$$\lim_{\text{vol}(\mathcal{D}) \rightarrow 0} \left[\frac{1}{\text{vol}(\mathcal{D})} \int_{\partial \mathcal{D}'} \frac{\partial f}{\partial n} d\sigma \right]$$

contains directional derivatives only in directions orthogonal to the radial direction. This term thus contains no derivatives with respect to r and only derivatives with respect to the angles $\phi, \theta_1, \theta_2, \dots, \theta_{p-2}$; we will indicate it by

$$\frac{1}{r^2} \Delta_{S^{p-1}} f,$$

where we determined the prefactor $1/r^2$ through dimensional analysis. We call the operator $\Delta_{S^{p-1}}$ the *spherical Laplace operator in $p - 1$ dimensions*. Equation (2.13) thus implies the following proposition.

Proposition 2.5

$$\Delta_p = \frac{\partial^2}{\partial r^2} + \frac{p-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{p-1}}.$$

Now that we are comfortable working in \mathbb{R}^p let us move on to briefly study the subject of orthogonal polynomials which will be very useful when we come to our main topic of discussion.

Chapter 3

Orthogonal Polynomials

This chapter provides the reader with several useful facts from the theory of orthogonal polynomials that we will require in order to prove some of the theorems in our main discussion. In what follows, we will assume the reader has taken an introductory course in linear algebra and, in particular, is familiar with inner product spaces and the Gram-Schmidt orthogonalization procedure. If this is not the case, we refer him or her to [4].

3.1 Orthogonality and Expansions

We will deal with a set \mathcal{P} where the members are real polynomials of finite degrees in one variable defined on some interval¹ I . With vector addition defined as the usual addition of polynomials and scalar multiplication defined as ordinary multiplication by real numbers, it is easy to check that we have, so far, a well defined vector space.

Let us put more structure on this space by adding an inner product. For any $p, q \in \mathcal{P}$, define a function $\langle \cdot, \cdot \rangle_w : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ by

$$\langle p, q \rangle_w \stackrel{\text{def}}{=} \int_I p(x)q(x)w(x) dx, \quad (3.1)$$

where w , called a *weight function*, is some positive function defined on I such that

$$\int_I r(x)w(x) dx$$

exists and is finite for all polynomials $r \in \mathcal{P}$. Notice that, since the product of any two polynomials is itself a polynomial, the above requirement ensures

¹For now, we allow I to be finite or infinite and open, closed, or neither.

the existence of $\langle p, q \rangle_w$ for all polynomials p, q . Moreover, the function (3.1) is symmetric and linear in both arguments and, since w is positive, $\langle r, r \rangle_w \geq 0$ for all polynomials r . Finally, since r^2 is continuous for all polynomials r ,

$$\langle r, r \rangle_w = \int_I r(x)^2 w(x) dx = 0 \quad \text{if and only if} \quad r(x) = 0 \text{ for all } x \in I.$$

For suppose $r(x_0) \neq 0$ for some $x_0 \in I$. By continuity, r must not vanish in some neighborhood about x_0 , and thus the above integral will acquire a positive value. Therefore, \mathcal{P} together with the function (3.1) is a well defined inner product space. We call $\langle \cdot, \cdot \rangle_w$ the *inner product with respect to the weight w* .

The reader should recall the following two inequalities that hold in any inner product space. We state them here for reference without any proofs which may be found in [4].

Proposition 3.1 (Cauchy-Schwarz Inequality) *For any vectors p, q in an inner product space \mathcal{P} , we have $|\langle p, q \rangle| \leq \|p\| \|q\|$.*

Proposition 3.2 (Triangle Inequality)² *For any vectors p, q in an inner product space \mathcal{P} , we have $\|p + q\| \leq \|p\| + \|q\|$.*

Now that we have an inner product space, we can speak of orthogonal polynomials. We call two polynomials p, q *orthogonal with respect to the weight*³ w provided $\langle p, q \rangle_w = 0$.

Suppose now that we have a basis $\mathcal{B} = \{\phi_n\}_{n=0}^\infty$ for \mathcal{P} consisting of orthogonal polynomials where ϕ_n is of degree n . We can easily see that such a basis exists by beginning with a set of monomials $\{x^n\}_{n=0}^\infty$ and, using the Gram-Schmidt process, to come up with the $\{\phi_n\}_{n=0}^\infty$: We leave the first member $x^0 = 1$ of the basis as is. That is, $\phi_0 = 1$. Then, from the second member $x^1 = x$, we subtract its component along the first member to get $\phi_1 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$, so that the second member of our new basis is orthogonal to the first. We continue in this fashion, at each step subtracting from x^n its components along the first $n - 1$ orthogonalized basis members. This creates an orthogonal basis for \mathcal{P} .

Now since \mathcal{B} is a basis, given any polynomial $q \in \mathcal{P}$ of degree k we can write⁴, for some scalars c_n ,

$$q = \sum_{n=0}^{\infty} c_n \phi_n, \quad (3.2)$$

²This is also called the Minkowski inequality.

³Although the weight function is important, often we tend not to stress it and call the two polynomials simply *orthogonal*. We also use the simplified notation $\langle \cdot, \cdot \rangle$.

⁴We do not need to worry about the convergence of the following series — we will see in a moment that it is actually a finite sum.

i.e., we can expand q in terms of the ϕ_n . Since each polynomial ϕ_n in \mathcal{B} has degree n , we can clearly write any q in terms of the set $\mathcal{B}_k = \{\phi_n\}_{n=0}^k$. Indeed, the space of all polynomials of degree k or less has dimension $k + 1$, and the linearly independent set \mathcal{B}_k has $k + 1$ elements. Using the uniqueness of the expansion (3.2), we can see that

$$c_n = 0 \text{ for all } n > k = \deg q. \quad (3.3)$$

Since the coefficients are given by

$$c_n = \frac{\langle q, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle},$$

we have the following result.

Proposition 3.3 *Given any polynomial q and any set of orthogonal polynomials $\{\phi_n\}_{n=0}^\infty$, where ϕ_n has degree n ,*

$$\langle \phi_n, q \rangle_w = \int_I \phi_n q w \, dx = 0, \text{ for all } n > \deg q.$$

Since we will come across expressions of the form $\langle p, p \rangle$ frequently, let us define the *norm of p with respect to the weight $w(x)$* as

$$\|p\|_w \stackrel{\text{def}}{=} \sqrt{\langle p, p \rangle_w}.$$

Keeping in mind the way we constructed our set of orthogonal polynomials $\{\phi_n\}_{n=0}^\infty$ (requiring ϕ_0 to have degree 0 — i.e., requiring ϕ_0 to be the constant polynomial $\phi_0 = a_0$ — and choosing each successive ϕ_n to be of degree n and orthogonal to the previous n members) the following result should not surprise the reader.

Proposition 3.4 *Any two orthogonal bases of polynomials $\{\phi_n\}_{n=0}^\infty$ and $\{\psi_n\}_{n=0}^\infty$ for which $\deg \phi_n = \deg \psi_n = n$, must be the same up to some nonzero multiplicative factors.*

Proof Since the $\{\phi_n\}_{n=0}^\infty$ constitute a basis, we can write any ψ_m as, using (3.3),

$$\psi_m = \sum_{n=0}^m c_n \phi_n$$

for some constants c_n given by

$$c_n = \frac{1}{\|\phi_n\|^2} \langle \psi_m, \phi_n \rangle.$$

By Proposition 3.3, $c_n = 0$ for all $n \neq m$. We are thus left with

$$\phi_m = c_m \psi_m,$$

and we are done. ■

3.2 The Recurrence Formula

We can find a useful recursive relationship between any three consecutive orthogonal polynomials $\phi_{n+1}, \phi_n, \phi_{n-1}$. In what follows, we will let k_n, ℓ_n denote the coefficients of the x^n term and the x^{n-1} term in ϕ_n , respectively.

Proposition 3.5 *Given any set of orthogonal polynomials $\mathcal{B} = \{\phi_n\}_{n=0}^\infty$, where ϕ_n has degree n ,*

$$\phi_{n+1} - (A_n x + B_n)\phi_n + C_n \phi_{n-1} = 0, \quad (3.4)$$

where

$$A_n = \frac{k_{n+1}}{k_n}, \quad B_n = A_n \left(\frac{\ell_{n+1}}{k_{n+1}} - \frac{\ell_n}{k_n} \right), \quad C_n = \frac{A_n}{A_{n-1}} \frac{\|\phi_n\|^2}{\|\phi_{n-1}\|^2}, \quad (3.5)$$

and $C_0 = 0$.

Proof From the formula given for A_n in (3.5), we see that $\phi_{n+1} - A_n x \phi_n$ must be a polynomial of degree n since the leading term of ϕ_{n+1} is canceled. Since the set \mathcal{B} is a basis, we can thus write

$$\phi_{n+1} - A_n x \phi_n = \sum_{j=0}^n \gamma_j \phi_j,$$

using (3.3). Taking the inner product of each side with $\phi_{j'}$, where $0 \leq j' \leq n$, we find

$$\gamma_{j'} = \frac{1}{\|\phi_{j'}\|^2} (\langle \phi_{n+1}, \phi_{j'} \rangle - A_n \langle x \phi_n, \phi_{j'} \rangle),$$

using the orthogonality of \mathcal{B} and the linearity of the inner product. From the definition of the inner product (3.1), we see that the above equation is the same as

$$\gamma_{j'} = \frac{1}{\|\phi_{j'}\|^2} (\langle \phi_{n+1}, \phi_{j'} \rangle - A_n \langle \phi_n, x \phi_{j'} \rangle).$$

Since $j' \neq n+1$, the first inner product in the above equation vanishes. Since $\deg(x \phi_{j'}) = \deg(\phi_{j'}) + 1 = j' + 1$, we can use Proposition 3.3 to determine

that the second inner product above is zero unless $j' = n - 1$ or $j' = n$. If we rename $\gamma_n = B_n$ and $\gamma_{n-1} = -C_n$, we have

$$\phi_{n+1} - (A_n x + B_n)\phi_n + C_n\phi_{n-1} = 0,$$

as in (3.4), with

$$B_n = \frac{-A_n}{\|\phi_n\|^2} \langle \phi_n, x\phi_n \rangle, \quad C_n = \frac{A_n}{\|\phi_{n-1}\|^2} \langle \phi_n, x\phi_{n-1} \rangle, \quad (3.6)$$

and it remains to compute the coefficients B_n, C_n . We will start with C_n , rewriting $x\phi_{n-1} = x(k_{n-1}x^{n-1} + \dots)$ as

$$\begin{aligned} k_{n-1}x^n + \text{lower order terms} &= \frac{k_{n-1}}{k_n} (k_n x^n + \text{lower order terms}) \\ &= \frac{k_{n-1}}{k_n} (\phi_n + \text{lower order terms}). \end{aligned}$$

Since the lower order terms ‘die’ in the inner product with ϕ_n , (3.6) implies

$$C_n = \frac{A_n}{\|\phi_{n-1}\|^2} \frac{k_{n-1}}{k_n} \|\phi_n\|^2 = \frac{A_n}{A_{n-1}} \frac{\|\phi_n\|^2}{\|\phi_{n-1}\|^2},$$

as required. We calculate B_n in a similar way, rewriting $x\phi_n = k_n x^{n+1} + \ell_n x^n + \dots$ as

$$\begin{aligned} x\phi_n &= \frac{k_n}{k_{n+1}} \left[k_{n+1}x^{n+1} + \ell_{n+1}x^n - \left(\ell_{n+1} - \frac{k_{n+1}\ell_n}{k_n} \right) x^n + \text{L.O.T.} \right] \\ &= \frac{k_n}{k_{n+1}} \phi_{n+1} + \left(\frac{\ell_n}{k_n} - \frac{\ell_{n+1}}{k_{n+1}} \right) k_n x^n + \text{L.O.T.} \\ &= \frac{k_n}{k_{n+1}} \phi_{n+1} + \left(\frac{\ell_n}{k_n} - \frac{\ell_{n+1}}{k_{n+1}} \right) \phi_n + \text{L.O.T.} \end{aligned}$$

where L.O.T. is an acronym which stands for “lower order terms”. Only the ϕ_n term will survive the inner product with ϕ_n , and (3.6) implies

$$B_n = \frac{-A_n}{\|\phi_n\|^2} \left(\frac{\ell_n}{k_n} - \frac{\ell_{n+1}}{k_{n+1}} \right) \|\phi_n\|^2 = A_n \left(\frac{\ell_{n+1}}{k_{n+1}} - \frac{\ell_n}{k_n} \right),$$

and we are done. ■

We see that if we know the coefficients A_n, B_n, C_n then once we have chosen two consecutive members of our set of orthogonal polynomials, the rest are determined.

3.3 The Rodrigues Formula

Until now, we spoke of the arbitrary interval I . For the remainder of our discussion, we will be most concerned with the interval $[-1, 1]$.

Consider the functions $\psi_n(x)$ defined on $[-1, 1]$ for $n = 0, 1, \dots$ as

$$\psi_n(x) = \frac{1}{w(x)} \left(\frac{d}{dx} \right)^n [w(x)(1-x^2)^n]. \quad (3.7)$$

For arbitrary choice of the weight function w , we cannot say whether these functions are polynomials nor whether they are orthogonal with respect to w . In an effort to force the ψ_n to be polynomials, we can start by looking at ψ_1 and requiring it to be a polynomial of degree 1,

$$\psi_1(x) = \frac{w'(x)}{w(x)}(1-x^2) - 2x \stackrel{\text{req}}{=} ax + b.$$

This creates a differential equation, which we can write as

$$\frac{w'}{w} = \frac{(a+2)x + b}{(1+x)(1-x)}.$$

Integrating,

$$\ln w = \int \frac{(a+2)x + b}{(1+x)(1-x)} dx + \text{const.}$$

Decomposing the integrand into partial fractions to compute the integral, we find

$$\ln w = -\frac{a+b+2}{2} \ln(1-x) + \frac{b-a-2}{2} \ln(1+x) + \text{const.},$$

or,

$$w(x) = \text{const.} \times (1-x)^\alpha (1+x)^\beta,$$

where α, β are the coefficients from the previous expression. Clearly, the arbitrary constant in the above expression cancels when $w(x)$ is inserted into (3.7), so we will take it to be unity. We will also require $\alpha, \beta > -1$ so that we can integrate any polynomial with respect to this weight. For our purposes, we will see that these restrictions on $w(x)$ are all we need.

Proposition 3.6 *Let $w(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha, \beta > -1$. Then (3.7) defines a set of polynomials $\{\psi_n\}_{n=0}^\infty$ where $\deg \psi_n = n$.*

In what follows, we will use $(k)_\ell$ to denote the *falling factorial*,

$$\begin{aligned} (k)_\ell &\stackrel{\text{def}}{=} k(k-1) \cdots (k-\ell+1) \text{ for each } \ell \in \mathbb{N}, \\ (k)_0 &\stackrel{\text{def}}{=} 1, \end{aligned}$$

and

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!}.$$

Proof We can see this using the *Leibnitz rule* for differentiating products,

$$\left(\frac{d}{dx}\right)^n (fg) = \sum_{k=0}^n \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}},$$

which can be proved easily by induction.

With the weight $w(x)$ given previously, (3.7) becomes

$$\begin{aligned} \psi_n(x) &= (1-x)^{-\alpha}(1+x)^{-\beta} \left(\frac{d}{dx}\right)^n \left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \\ &= \sum_{k=0}^n \binom{n}{k} \left[(1-x)^{-\alpha} \left(\frac{d}{dx}\right)^k (1-x)^{n+\alpha}\right] \left[(1+x)^{-\beta} \left(\frac{d}{dx}\right)^{n-k} (1+x)^{n+\beta}\right] \\ &= \sum_{k=0}^n \binom{n}{k} \left[(-1)^k (n+\alpha)_k (1-x)^{n-k}\right] \left[(n+\beta)_{n-k} (1+x)^k\right]. \end{aligned}$$

We can see that ψ_n is a polynomial. Let us examine the leading term in ψ_n . We see from the last line in the above equation that

$$\begin{aligned} \psi_n(x) &= \sum_{k=0}^n \binom{n}{k} (-1)^k (n+\alpha)_k (-x)^{n-k} (n+\beta)_{n-k} x^k + \text{L.O.T.} \\ &= (-1)^n x^n \sum_{k=0}^n \binom{n}{k} (n+\alpha)_k (n+\beta)_{n-k} + \text{L.O.T.} \end{aligned}$$

Since the sum in the last line of the above equation is strictly positive, we see that the x^n term in ψ_n has a nonvanishing coefficient; i.e., the degree of ψ_n is precisely n . ■

Proposition 3.7 *Let $w(x) = (1-x)^\alpha(1+x)^\beta$, for $\alpha, \beta > -1$. Then the ψ_n defined in (3.7) satisfy*

$$\int_{-1}^1 \psi_n(x) x^k w(x) dx = 0, \quad \text{for } 0 \leq k < n.$$

Proof Let J be the above integral. Inserting (3.7) for ψ_n and integrating by parts n times,

$$\begin{aligned}
 J &= \int_{-1}^1 x^k \left(\frac{d}{dx} \right)^n [w(x)(1-x^2)^n] dx \\
 &= x^k \left(\frac{d}{dx} \right)^{n-1} [w(x)(1-x^2)^n] \Big|_{-1}^1 - \int_{-1}^1 kx^{k-1} \left(\frac{d}{dx} \right)^{n-1} [w(x)(1-x^2)^n] dx \\
 &\vdots \\
 &= \sum_{\ell=0}^k (-1)^\ell (k)_\ell x^{k-\ell} \left(\frac{d}{dx} \right)^{n-\ell-1} [w(x)(1-x^2)^n] \Big|_{-1}^1.
 \end{aligned}$$

Upon inserting the assumed form of $w(x)$, this becomes

$$J = \sum_{\ell=0}^k (-1)^\ell (k)_\ell x^{k-\ell} \left(\frac{d}{dx} \right)^{n-\ell-1} \left[(1-x)^{n+\alpha} (1+x)^{n+\beta} \right] \Big|_{-1}^1.$$

In each term of the above sum, the factors $(1-x)^{n+\alpha}$ and $(1+x)^{n+\beta}$ keep positive exponents after being operated upon by $\left(\frac{d}{dx}\right)^{n-\ell-1}$ so that J vanishes after we evaluate the sum at -1 and 1 . ■

Since each ψ_n has degree n and is orthogonal to all polynomials with degree less than n , we have the following result.

Corollary 3.8 *The formula (3.7) defines an orthogonal set of polynomials $\{\psi_n\}_{n=0}^\infty$ known as the Jacobi polynomials with $\deg \psi_n = n$ for each n .*

We have thus found that (3.7), called the *Rodrigues formula*, with an acceptable weight function $w(x)$, defines a set of orthogonal polynomials. We are free to multiply each polynomial ψ_n by an arbitrary constant without upsetting this fact. The different classical orthogonal polynomials can be defined using the Rodrigues formula by adjusting this constant and the exponents α, β in $w(x)$.

Now, suppose we discover that a set of polynomials $\{\phi_n\}_{n=0}^\infty$ such that $\deg \phi_n = n$ is orthogonal with respect to the weight $w(x)$. Can we conclude that these polynomials satisfy the Rodrigues formula (3.7)? Comparing Corollary 3.8 and Proposition 3.4, we can answer “yes.” That is, we can define the ϕ_n by the Rodrigues formula with suitable multiplicative constants,

$$\phi_n(x) = \frac{c_n}{w(x)} \left(\frac{d}{dx} \right)^n [w(x)(1-x^2)^n].$$

3.4 Approximations by Polynomials

The main result in this section will be the Weierstrass approximation theorem, which will allow us to use polynomials to approximate continuous functions on closed intervals. First, we will introduce some ideas that we will use to prove this important theorem.

Given a function $f : [0, 1] \rightarrow \mathbb{R}$, we will use $B_n(x; f)$ to denote a *Bernstein polynomial*, defined on the closed interval $[0, 1]$ as

$$B_n(x; f) \stackrel{\text{def}}{=} \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

for $n = 1, 2, \dots$. Since $f\left(\frac{k}{n}\right)$ is just a constant, these are clearly polynomials.

We will compute three special cases of the Bernstein polynomials.

Lemma 3.9 *For every natural number n ,*

$$B_n(x; 1) = 1, \tag{3.8}$$

$$B_n(x; x) = x, \tag{3.9}$$

$$B_n(x; x^2) = x^2 + \frac{x(1-x)}{n}. \tag{3.10}$$

Proof By the binomial theorem,

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}. \tag{3.11}$$

Substituting $y = 1 - x$ gives

$$1 = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = B_n(x; 1), \tag{3.12}$$

proving (3.8). Differentiating (3.11) with respect to x ,

$$n(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^{k-1} y^{n-k},$$

and multiplying by x/n ,

$$x(x+y)^{n-1} = \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right) x^k y^{n-k}. \tag{3.13}$$

Substituting $y = 1 - x$ gives

$$x = \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right) x^k (1-x)^{n-k} = B_n(x; x),$$

proving (3.9). Differentiating (3.13) with respect to x ,

$$(x+y)^{n-1} + (n-1)x(x+y)^{n-2} = \sum_{k=0}^n \binom{n}{k} \left(\frac{k^2}{n}\right) x^{k-1} y^{n-k},$$

and multiplying by x/n ,

$$\left(x^2 + \frac{xy}{n}\right) (x+y)^{n-2} = \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right)^2 x^k y^{n-k}.$$

Substituting $y = 1 - x$ gives

$$x^2 + \frac{x(1-x)}{n} = \sum_{k=0}^n \binom{n}{k} \left(\frac{k}{n}\right)^2 x^k (1-x)^{n-k} = B_n(x; x^2),$$

proving (3.10). ■

We are now well equipped to prove the following result.

Theorem 3.10 (Weierstrass Approximation Theorem) *Let the function $f : [a, b] \rightarrow \mathbb{R}$ be continuous, and let $\epsilon > 0$. Then there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all x in the interval $[a, b]$.*

We will show that the Bernstein polynomials, for sufficiently large n , work to approximate the function f to the required accuracy.

Proof Let $\epsilon > 0$. First, we will redefine the function f so that we can work in the interval $[0, 1]$. Using the linear transformation

$$T(x) = (b-a)x + a,$$

we set

$$\tilde{f} = f \circ T : [0, 1] \rightarrow \mathbb{R}.$$

If we can find a polynomial p to adequately approximate \tilde{f} , we can use the polynomial $p \circ T^{-1} : [a, b] \rightarrow \mathbb{R}$ as our required approximation of f . From here on, we will write f instead of \tilde{f} for simplicity.

Since f is continuous on the closed and bounded interval $[0, 1]$, we know that f is bounded and uniformly continuous. By the definition of boundedness, there exists an M such that

$$|f(x)| < M, \text{ for every } x \in [0, 1],$$

which implies

$$|f(x) - f\left(\frac{k}{n}\right)| \leq |f(x)| + \left|f\left(\frac{k}{n}\right)\right| < 2M, \text{ for all } x \in [0, 1] \text{ and } 0 \leq k \leq n. \quad (3.14)$$

By the definition of uniform continuity, there exists a $\delta > 0$ such that

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2}, \text{ whenever } \left|x - \frac{k}{n}\right| < \delta. \quad (3.15)$$

Now, let

$$E = |f(x) - B_n(x; f)|,$$

and we will estimate this error from above. Using (3.8) and the triangle inequality,

$$\begin{aligned} E &= \left|f(x) - \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}\right| \\ &= \left|\sum_{k=0}^n \binom{n}{k} \left[f(x) - f\left(\frac{k}{n}\right)\right] x^k (1-x)^{n-k}\right| \\ &\leq \sum_{k=0}^n \binom{n}{k} \left|f(x) - f\left(\frac{k}{n}\right)\right| x^k (1-x)^{n-k}. \end{aligned}$$

Splitting up the sum,

$$\begin{aligned} E &\leq \sum_{\left|x - \frac{k}{n}\right| < \delta} \binom{n}{k} \left|f(x) - f\left(\frac{k}{n}\right)\right| x^k (1-x)^{n-k} \\ &\quad + \sum_{\left|x - \frac{k}{n}\right| \geq \delta} \binom{n}{k} \left|f(x) - f\left(\frac{k}{n}\right)\right| x^k (1-x)^{n-k}. \end{aligned}$$

Using (3.14) and (3.15),

$$\begin{aligned} E &< \frac{\epsilon}{2} \sum_{\left|x - \frac{k}{n}\right| < \delta} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{\left|x - \frac{k}{n}\right| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{\epsilon}{2} \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} + \frac{2M}{\delta^2} \sum_{k=0}^n \binom{n}{k} \left(x - \frac{k}{n}\right)^2 x^k (1-x)^{n-k}, \end{aligned}$$

or

$$E < \frac{\epsilon}{2} + \frac{2M}{\delta^2} [x^2 B_n(x; 1) - 2x B_n(x; x) + B_n(x; x^2)]$$

Using Lemma 3.9,

$$E < \frac{\epsilon}{2} + \frac{2M}{\delta^2} \left(x^2 - 2x^2 + x^2 + \frac{x(1-x)}{n} \right).$$

Since the function $x(1-x)$ has a maximum value of $1/4$, we can write

$$E < \frac{\epsilon}{2} + \frac{2M}{4n\delta^2} \leq \epsilon, \text{ for all } n \geq \frac{M}{\epsilon\delta^2}.$$

Thus, we see that the Bernstein polynomials do the job for large enough n , completing the proof. ■

Now that we know polynomials are a good tool for approximating continuous functions, we ask how to find the best such approximation. This is a result from linear algebra. For a function f , we define the best approximation p to be the one that minimizes the norm of the error, i.e., for which $\|f - p\|$ is smallest.

Proposition 3.11 *Let f be a function, and let $\{\phi_k\}_{k=0}^\infty$ be an orthogonal set of polynomials with $\deg \phi_n = n$. The polynomial*

$$p_n = \sum_{k=0}^n a_k \phi_k, \quad \text{where} \quad a_k = \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2}, \quad (3.16)$$

is the unique polynomial of degree n that best approximates f , i.e., that minimizes $\|f - p_n\|$.

Proof Let's choose a different polynomial \tilde{p}_n of degree n . We will write \tilde{p}_n as

$$\tilde{p}_n = p_n + q_n, \quad \text{where} \quad q_n = \sum_{k=0}^n b_k \phi_k,$$

for some constants b_k not all zero. Then, we can write $\|f - \tilde{p}_n\|^2$ as

$$\|f - p_n - q_n\|^2 = \|f - p_n\|^2 + \|q_n\|^2 - 2\langle f - p_n, q_n \rangle.$$

But

$$\langle f - p_n, q_n \rangle = \sum_{k=0}^n b_k (\langle f, \phi_k \rangle - \langle p_n, \phi_k \rangle) = 0,$$

since each term in parentheses vanishes, as we can see from (3.16). Thus, for any \tilde{p}_n ,

$$\|f - \tilde{p}_n\|^2 = \|f - p_n\|^2 + \|q_n\|^2 > \|f - p_n\|^2,$$

the error is greater than that of p_n . ■

This fact is really just a special case of a more general result from the study of inner product spaces in linear algebra. We will state the more general fact next. The proof is exactly the same.

Proposition 3.12 *Let f be a vector in an inner product space, and let $\{\phi_k\}_{k=0}^{\infty}$ be an orthogonal set of basis vectors. The vector*

$$p_n = \sum_{k=0}^n a_k \phi_k, \quad \text{where} \quad a_k = \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2},$$

is the unique linear combination of the first $n + 1$ basis vectors that best approximates f , i.e., that minimizes $\|f - p_n\|$.

3.5 Hilbert Space and Completeness

Later in the main discussion, we will be interested in expanding an arbitrary function f in an infinite series of the spherical harmonic functions. In this section, we will address the question of when such an expansion is possible. Since we are concerned primarily with spherical harmonics in an arbitrary number of dimensions, we will keep the discussion general, making no reference here to the number of variables on which the functions depend. In what follows we will thus let x denote a vector in \mathbb{R}^p , \mathcal{D} denote a subset of \mathbb{R}^p , and $d\Omega$ denote the differential volume element of \mathcal{D} .

We will restrict our attention to the space of square-integrable functions with domain \mathcal{D} with respect to the weight $w(x)$, i.e., those functions $f : \mathcal{D} \rightarrow \mathbb{R}$ for which the integral

$$\int_{\mathcal{D}} f(x)^2 w(x) d\Omega$$

exists and is finite. When endowed with the usual operations of function addition and multiplication by real numbers and paired with the inner product

$$\langle f, g \rangle = \int_{\mathcal{D}} f(x)g(x)w(x) d\Omega,$$

the set of such functions becomes an inner product space. Such a space has an induced norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

We will now work up to a definition of a Hilbert space.

Definition In a normed space, a sequence $\{x_n\}_{n=0}^{\infty}$ is called *Cauchy* provided that for any $\epsilon > 0$ there exists an N such that

$$\|x_n - x_m\| < \epsilon \text{ for all } n, m \geq N.$$

A Cauchy sequence is always convergent. However, it is possible that its limit does not belong to the same space.

Definition A normed space is *complete* provided every Cauchy sequence converges to an element in the space.

Definition A complete inner product space is called a *Hilbert space*.

We will state the following fact without proof. The interested reader may consult [9] for the proof.

Theorem 3.13 *The inner product space of square-integrable functions defined above is a Hilbert space.*

We can restate the question that opened this section as follows. Given a Hilbert space \mathcal{H} and an *orthonormal* set $\{\phi_n\}_{n=0}^\infty \subseteq \mathcal{H}$ (which can be obtained from any orthogonal set by dividing each member by its norm so $\|\phi_n\| = 1$), when can we write any arbitrary member $f \in \mathcal{H}$ as a linear combination⁵ of the ϕ_n ?

Definition An orthonormal set $\{\phi_n\}_{n=0}^\infty \subseteq \mathcal{H}$ is called *complete* provided that for each $f \in \mathcal{H}$, there exist scalars c_1, c_2, \dots , such that

$$\lim_{n \rightarrow \infty} \|f - \sum_{k=0}^n c_k \phi_k\| = 0. \quad (3.17)$$

We know from Proposition 3.12 that out of all linear combinations, the combination

$$p_n = \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k$$

minimizes $\|f - p_n\|$. Notice that we have now normalized the vectors ϕ_k to unity: $\|\phi_k\| = 1$. Thus, if there exist scalars c_k for which the norm in (3.17) converges to zero, then certainly

$$\lim_{n \rightarrow \infty} \|f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k\| = 0, \quad (3.18)$$

since

$$\|f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k\| \leq \|f - \sum_{k=0}^n c_k \phi_k\|, \text{ for every } n.$$

⁵A linear combination is usually assumed to contain a finite number of terms. We will not need this restriction for our purposes. Here, linear combinations can be finite or infinite sums.

Let us rewrite (3.18) by computing

$$\left\| f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k \right\|^2,$$

which is the same as

$$\|f\|^2 - 2 \left\langle f, \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k \right\rangle + \left\langle \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k, \sum_{\ell=0}^n \langle f, \phi_\ell \rangle \phi_\ell \right\rangle.$$

Using the linearity of the inner product, this becomes

$$\|f\|^2 - 2 \sum_{k=0}^n \langle f, \phi_k \rangle^2 + \sum_{k,\ell=0}^n \langle f, \phi_k \rangle \langle f, \phi_\ell \rangle \langle \phi_k, \phi_\ell \rangle.$$

By the orthonormality of the ϕ_n , we have

$$\left\| f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k \right\|^2 = \|f\|^2 - \sum_{k=0}^n \langle f, \phi_k \rangle^2. \quad (3.19)$$

We note that since the norm-squared is non-negative, then

$$\|f\|^2 - \sum_{k=0}^n \langle f, \phi_k \rangle^2 \geq 0,$$

or,

$$\sum_{k=0}^n \langle f, \phi_k \rangle^2 \leq \|f\|^2, \text{ for every } n. \quad (3.20)$$

This is known as *Bessel's inequality* and tells us that the sum $\sum_{k=0}^{\infty} \langle f, \phi_k \rangle^2$ converges. If the ϕ_k form a complete set, $\|f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k\|^2$ must converge to zero as $n \rightarrow \infty$. In this case, from equation (3.19), it follows that Bessel's inequality becomes an equality:

$$\sum_{k=0}^{\infty} \langle f, \phi_k \rangle^2 = \|f\|^2,$$

known as *Parseval's equality*. We have thus arrived at the following conclusion.

Proposition 3.14 *An orthonormal set $\{\phi_n\}_{n=0}^{\infty} \subseteq \mathcal{H}$ is complete if and only if Parseval's equality holds for each $f \in \mathcal{H}$.*

We give one more definition.

Definition A set $\{\phi_n\}_{n=0}^\infty$ is *closed* provided

$$\langle f, \phi_n \rangle = 0 \text{ for every } n \quad \text{implies} \quad f = 0.$$

Now we are ready to prove the main result of this section.

Theorem 3.15 *The orthonormal set $\{\phi_n\}_{n=0}^\infty \subseteq \mathcal{H}$ is complete if and only if it is closed.*

Proof (Sufficient) Let $\{\phi_n\}_{n=0}^\infty \subset \mathcal{H}$ constitute a closed orthonormal set and $f \in \mathcal{H}$. We will show that

$$\lim_{n \rightarrow \infty} \left(\|f\|^2 - \sum_{k=0}^n \langle f, \phi_k \rangle^2 \right) = 0,$$

since this guarantees completeness by Proposition 3.14.

Define $g_n = f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k$, and notice that $\{g_n\}_{n=0}^\infty$ is a Cauchy sequence since, if $n > m$,

$$\begin{aligned} \|g_n - g_m\|^2 &= \left\| \sum_{k=m+1}^n \langle f, \phi_k \rangle \phi_k \right\|^2 \\ &= \sum_{k, \ell=m+1}^n \langle f, \phi_k \rangle \langle f, \phi_\ell \rangle \langle \phi_k, \phi_\ell \rangle \\ &= \sum_{k=m+1}^n \langle f, \phi_k \rangle^2, \end{aligned}$$

which can be made arbitrarily small for large enough n, m since the series $\sum_{k=0}^\infty \langle f, \phi_k \rangle^2$ converges, by (3.20). Since \mathcal{H} is complete by definition, this Cauchy sequence must converge to some $g \in \mathcal{H}$, i.e.,

$$\lim_{n \rightarrow \infty} \|g_n - g\| = 0. \quad (3.21)$$

Now fix j and take $n \in \mathbb{N}_0$ such that $n > j$. By the Cauchy-Schwartz inequality

$$|\langle g, \phi_j \rangle| = |\langle g_n - g, \phi_j \rangle| \leq \|g_n - g\| \|\phi_j\| = \|g_n - g\|.$$

Since this holds for any $n > j$, we can use (3.21) to conclude

$$|\langle g, \phi_j \rangle| \leq \lim_{n \rightarrow \infty} \|g_n - g\| = 0,$$

which implies that for arbitrary j , $\langle g, \phi_j \rangle = 0$. Since we assumed the ϕ_j constitute a closed set, we have $g = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \left(\|f\|^2 - \sum_{k=0}^n \langle f, \phi_k \rangle^2 \right) = \lim_{n \rightarrow \infty} \|g_n\|^2 = \lim_{n \rightarrow \infty} \|g_n - g\|^2 = 0.$$

(Necessary) Suppose $\{\phi_n\}_{n=0}^\infty$ is complete but not closed. Then there exists a function $f \neq 0$ such that $\langle f, \phi_n \rangle = 0$ for every n . Then

$$\lim_{n \rightarrow \infty} \left(\|f\|^2 - \sum_{k=0}^n \langle f, \phi_k \rangle^2 \right) = \|f\|^2 \neq 0.$$

So, by Proposition 3.14, $\{\phi_n\}_{n=0}^\infty$ is not complete — a contradiction. This proves that if an orthonormal set in \mathcal{H} is closed, then it is complete. ■

In Chapter 4, we will use this result to show that the set of spherical harmonics form a complete set in the space of square-integrable functions by showing that it is closed.

Chapter 4

Spherical Harmonics in p Dimensions

We will begin by developing some facts about a special kind of polynomials. We will then define a spherical harmonic to be one of these polynomials with a restricted domain, as hinted at in Section 1.1. After discussing some properties of spherical harmonics, we will introduce the Legendre polynomials. And once we have produced a considerable number of results, we will move on to an application of the material developed to boundary value problems.

4.1 Harmonic Homogeneous Polynomials

Definition A polynomial $H_n(x_1, x_2, \dots, x_p)$ is *homogeneous of degree n* in the p variables x_1, x_2, \dots, x_p provided

$$H_n(tx_1, tx_2, \dots, tx_p) = t^n H_n(x_1, x_2, \dots, x_p) .$$

In the definition of a homogeneous polynomial, let's set $u_i = tx_i$, for all i and differentiate the defining equation with respect to t :

$$\sum_{i=1}^p \frac{\partial H_n(u_1, u_2, \dots, u_p)}{\partial u_i} \frac{du_i}{dt} = n t^{n-1} H_n(x_1, x_2, \dots, x_p),$$

or

$$\sum_{i=1}^p \frac{\partial H_n(u_1, u_2, \dots, u_p)}{\partial u_i} x_i = n t^{n-1} H_n(x_1, x_2, \dots, x_p).$$

Finally, we set $t = 1$ to find the following functional equation satisfied by the homogeneous polynomial,

$$\sum_{i=1}^p \frac{\partial H_n(x_1, x_2, \dots, x_p)}{\partial x_i} x_i = n H_n(x_1, x_2, \dots, x_p) , \quad (4.1)$$

known as *Euler's equation*.

The following calculation will be useful in the counting of linearly independent homogeneous polynomials.

Lemma 4.1 For $0 < |r| < 1$,

$$\frac{1}{(1-r)^p} = \sum_{j=0}^{\infty} \frac{(p+j-1)!}{j!(p-1)!} r^j. \quad (4.2)$$

Proof We will use a counting trick to prove this. Since $0 < |r| < 1$, we can write $(1-r)^{-p}$ in terms of a product of geometric series, i.e.,

$$\begin{aligned} (1-r)^{-p} &= \underbrace{\left(\frac{1}{1-r}\right) \left(\frac{1}{1-r}\right) \cdots \left(\frac{1}{1-r}\right)}_{p \text{ times}} \\ &= \underbrace{\left(\sum_{n=0}^{\infty} r^n\right) \left(\sum_{n=0}^{\infty} r^n\right) \cdots \left(\sum_{n=0}^{\infty} r^n\right)}_{p \text{ times}}. \end{aligned}$$

We will compute the *Cauchy product* of the p infinite series. The result will be an infinite series including a constant term and all positive integer powers of r ,

$$(1-r)^{-p} = \sum_{j=0}^{\infty} c_j r^j.$$

It remains to compute the coefficients c_j . To determine each c_j , we must compute how many r^j 's are produced in the Cauchy product, i.e., in how many different ways we produce an r^j in the multiplication.

Note that this computation is equivalent to asking, "In how many different ways can we place j indistinguishable balls into p boxes?" The p boxes correspond to the p series in the product, and choosing to place $k < j$ balls into a certain box corresponds to choosing the r^k term in that series when computing a term of the Cauchy product.

Let us use a diagram to assist in this calculation. We will use a vertical line to denote a division between two boxes and a dot to denote a ball. Each configuration of the j balls in p boxes can thus be represented by a string of j dots and $p-1$ lines (since p boxes require only $p-1$ divisions). For example,

$$\bullet \bullet | \bullet | \bullet | | \bullet \bullet | \bullet | | \bullet$$

represents one configuration of $j = 8$ balls in $p = 8$ boxes.

Now, the number of ways to arrange j indistinguishable balls in p boxes is the same as the number of distinct arrangements of j dots and $p - 1$ lines. This is given by “ $p - 1 + j$ choose j ,” i.e.,

$$c_j = \binom{p-1+j}{j} = \frac{(p-1+j)!}{j!(p-1)!},$$

and the lemma is proved. ■

Proposition 4.2 *If $K(p, n)$ denotes the number of linearly independent homogeneous polynomials of degree n in p variables, then*

$$K(p, n) = \frac{(p+n-1)!}{n!(p-1)!}.$$

We will give two proofs of this claim. The first uses a recursive relation obeyed by $K(p, n)$, while the second employs another counting trick.

Proof 1 Let $H_n(x_1, x_2, \dots, x_p)$ be a homogeneous polynomial of degree n in its p variables. Notice that H_n is a polynomial in x_p of degree at most n . For if H_n contained a power of x_p greater than n , the polynomial could not be homogeneous of degree n , since $H_n(\dots, tx_p)$ would contain a power of t greater than n . Thus we can write,

$$H_n(x_1, x_2, \dots, x_p) = \sum_{j=0}^n x_p^j h_{n-j}(x_1, x_2, \dots, x_{p-1}), \quad (4.3)$$

where the h_{n-j} are polynomials. Moreover, notice that the h_{n-j} must be homogeneous of degree $n - j$ in their $p - 1$ variables. Indeed, using the homogeneity of H_n ,

$$t^n \sum_{j=0}^n x_p^j h_{n-j}(x_1, \dots, x_{p-1}) = H_n(tx_1, \dots, tx_p) = \sum_{j=0}^n (tx_p)^j h_{n-j}(tx_1, \dots, tx_{p-1}),$$

which implies that

$$\sum_{j=0}^n x_p^j [t^{n-j} h_{n-j}(x_1, \dots, x_{p-1}) - h_{n-j}(tx_1, \dots, tx_{p-1})] = 0.$$

The expression in brackets must vanish by the linear independence of the x_p^j . Each h_{n-j} can be written in terms of a basis of $K(p-1, n-j)$ homogeneous

polynomials of degree $n - j$ in $p - 1$ variables, and thus H_n can be written in terms of a basis of

$$K(p, n) = \sum_{j=0}^n K(p-1, n-j) = \sum_{j=0}^n K(p-1, j)$$

linearly independent elements. We have found a recursive relation that the $K(p, n)$ must satisfy. Now for some $0 < |r| < 1$, let

$$G(p) = \sum_{n=0}^{\infty} r^n K(p, n). \quad (4.4)$$

Then,

$$G(p) = \sum_{n=0}^{\infty} r^n \sum_{j=0}^n K(p-1, j) = \sum_{j=0}^{\infty} K(p-1, j) \sum_{n=j}^{\infty} r^n,$$

or

$$G(p) = \sum_{j=0}^{\infty} K(p-1, j) r^j \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \sum_{j=0}^{\infty} r^j K(p-1, j),$$

so

$$G(p) = \frac{G(p-1)}{1-r}.$$

Using an inductive argument, we can show that

$$G(p) = \frac{G(1)}{(1-r)^{p-1}}.$$

By noticing that $K(1, n) = 1$, since every homogeneous polynomial of degree n in one variable can be written as cx_1^n , we see that

$$G(1) = \sum_{j=0}^{\infty} r^j = \frac{1}{1-r}.$$

Thus,

$$G(p) = (1-r)^{-p} = \sum_{n=0}^{\infty} \frac{(p+n-1)!}{n!(p-1)!} r^n,$$

by the above lemma. Comparing this to (4.4) then proves the theorem. ■

Proof 2 Using reasoning similar to that used in the above proof, we see that every *monomial* (i.e., product of variables) in an n -th degree homogeneous polynomial must be of degree n . For example, a fourth degree homogeneous polynomial in the variables x, y can have an x^2y^2 but not an x^2y term.

To uniquely determine such a polynomial, we must give the coefficient of every possible n -th degree monomial. So to find $K(p, n)$, we must find out how many possible n -th degree monomials there are in p variables. But this is equivalent to asking, “In how many ways can we place n indistinguishable balls into p boxes?” The p boxes represent the p variables from which we can choose, and placing $j < n$ balls into a certain box corresponds to choosing to raise that variable to the j power. For example, using the same notation as in the proof of Lemma 4.1, the string

$$\bullet \bullet | \bullet | \bullet |$$

could represent the $w^2x^1y^1z^0$ term in a fourth-degree homogeneous polynomial. As in the above lemma, the number of such arrangements is

$$K(p, n) = \binom{n+p-1}{n} = \frac{(n+p-1)!}{n!(p-1)!},$$

proving the theorem. ■

Definition A polynomial $q(x_1, x_2, \dots, x_p)$ is *harmonic* provided

$$\Delta_p q = 0. \quad (4.5)$$

Equation (4.5) is called the *Laplace equation*.

The following property of combinations will be used to prove the theorem that follows it.

Lemma 4.3 *If $k, \ell \in \mathbb{N}$, then*

$$\binom{k}{\ell} = \frac{k}{\ell} \binom{k-1}{\ell-1}.$$

Proof Just compute:

$$\frac{k}{\ell} \binom{k-1}{\ell-1} = \frac{k}{\ell} \cdot \frac{(k-1)!}{(\ell-1)!(k-\ell)!} = \frac{k!}{\ell!(k-\ell)!} = \binom{k}{\ell},$$

as required. ■

Theorem 4.4 *If $N(p, n)$ denotes the number of linearly independent homogeneous harmonic polynomials of degree n in p variables, then*

$$N(p, n) = \frac{2n+p-2}{n} \binom{n+p-3}{n-1}.$$

Proof Let H_n be a homogeneous harmonic polynomial of degree n in p variables. As in (4.3), we write

$$H_n(x_1, x_2, \dots, x_p) = \sum_{j=0}^n x_p^j h_{n-j}(x_1, x_2, \dots, x_{p-1}),$$

and we operate on H_n with the Laplace operator. Since H_n is harmonic,

$$0 = \Delta_p H_n = \left(\frac{\partial^2}{\partial x_p^2} + \Delta_{p-1} \right) H_n = \sum_{j=2}^n j(j-1) x_p^{j-2} h_{n-j} + \sum_{j=0}^n x_p^j \Delta_{p-1} h_{n-j},$$

so

$$0 = \sum_{j=0}^n x_p^j [(j+2)(j+1)h_{n-j-2} + \Delta_{p-1} h_{n-j}].$$

where we define $h_{-1} = h_{-2} = 0$. Since the x_p^j are linearly independent, each of the coefficients must vanish,

$$\begin{aligned} 2h_{n-2} + \Delta_{p-1} h_n &= 0, \\ 6h_{n-3} + \Delta_{p-1} h_{n-1} &= 0, \\ &\vdots \\ n(n-1)h_0 + \Delta_{p-1} h_2 &= 0, \\ \Delta_{p-1} h_1 &= 0, \\ \Delta_{p-1} h_0 &= 0. \end{aligned} \tag{4.6}$$

We have found a recursive relationship that the h_{n-j} must obey. Thus, choosing h_n and h_{n-1} determines the rest of the h_{n-j} . By Theorem 4.2, h_n can be written in terms of $K(p-1, n)$ basis polynomials and h_{n-1} can be written in terms of $K(p-1, n-1)$ basis polynomials. Thus we must give $K(p-1, n) + K(p-1, n-1)$ coefficients to determine h_n, h_{n-1} which thereby determine H_n . Therefore,

$$N(p, n) = K(p-1, n) + K(p-1, n-1) = \binom{p+n-2}{n} + \binom{p+n-3}{n-1}.$$

By the above lemma, we can write this as

$$N(p, n) = \frac{p+n-2}{n} \binom{p+n-3}{n-1} + \binom{p+n-3}{n-1} = \frac{2n+p-2}{n} \binom{n+p-3}{n-1},$$

and thus the theorem is proved. ■

In what follows, x will always denote the vector (x_1, x_2, \dots, x_p) , r or $|x|$ will denote its norm $\sqrt{x_1^2 + x_2^2 + \dots + x_p^2}$, and ξ will denote the vector $(\xi_1, \xi_2, \dots, \xi_p)$ having unit norm. Keep in mind that x, ξ represent vectors while the x_j, ξ_j denote their components. Also, H_n will always denote a harmonic homogeneous polynomial. Lastly, in the remainder of this chapter, we will often suppress explicit reference to the number of dimensions in which we work. It should be assumed that we deal with p dimensions unless stated otherwise.

4.2 Spherical Harmonics and Orthogonality

We are now ready to introduce the spherical harmonics. First, notice that

$$H_n(x) = H_n(r\xi) = r^n H_n(\xi). \quad (4.7)$$

In p -dimensional spherical coordinates, the radial dependence and the angular dependence of the functions H_n can be separated.

Definition A *spherical harmonic of degree n* , denoted $Y_n(\xi)$, is a harmonic homogeneous polynomial of degree n in p variables restricted to the unit $(p-1)$ -sphere. In other words, Y_n is the map

$$Y_n : S^{p-1} \rightarrow \mathbb{R}, \text{ given by } Y_n(\xi) = H_n(\xi) \text{ for every } \xi \in S^{p-1}$$

for some harmonic homogeneous polynomial H_n . We can write $Y_n = H_n|_{S^{p-1}}$.

In Chapter 1, we introduced functions Y (which we claimed were spherical harmonics) that turned out to be eigenfunctions of the angular part of the Laplace operator. Using Proposition 2.5, we can show that spherical harmonics are indeed eigenfunctions of $\Delta_{S^{p-1}}$.

Proposition 4.5

$$\Delta_{S^{p-1}} Y_n = n(2 - p - n) Y_n. \quad (4.8)$$

Proof Let H_n be a harmonic homogeneous polynomial and Y_n its associated spherical harmonic. As in (4.7), we have $H_n(x) = r^n Y_n(\xi)$. Then, using Proposition 2.5,

$$0 = \Delta_p (r^n Y_n) = n(n-1)r^{n-2}Y_n + \frac{p-1}{r} n r^{n-1} Y_n + \frac{1}{r^2} r^n \Delta_{S^{p-1}} Y_n.$$

Rearranging,

$$r^{n-2} [\Delta_{S^{p-1}} Y_n + n(n+p-2)Y_n] = 0,$$

which implies

$$\Delta_{S^{p-1}} Y_n = n(2 - p - n) Y_n,$$

as sought. ■

Remark In three dimensions (4.8) becomes

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_\ell = -\ell(\ell + 1) Y_\ell.$$

As promised in Subsection 1.2, this allows us to show that three-dimensional spherical harmonics carry a definite amount of quantum mechanical angular momentum. Referring to (1.13),

$$\hat{L}^2 Y_\ell = \hbar^2 \ell(\ell + 1) Y_\ell.$$

Remark Since the spherical harmonic $Y_n(\xi)$ is defined as the restriction of some $H_n(x)$ to the unit sphere, $Y_n(\xi)$ must also be homogeneous. However, $Y_n(t\xi)$ is not always defined since the domain of the spherical harmonic is S^{p-1} . In fact, it is only defined for $|t\xi| = 1$, and since $|\xi| = 1$, we see that we can only write $Y_n(t\xi)$ for $t = \pm 1$. The case $t = 1$ is trivial, but the case $t = -1$ tells us the parity of $Y_n(\xi)$. Since $Y_n(-\xi) = (-1)^n Y_n(\xi)$, we see that the transformation $\xi \mapsto -\xi$ sends $Y_n \mapsto -Y_n$ if n is odd and leaves Y_n invariant if n is even.

We now come to the main result of this section, which we hinted at in Section 1.1.

Theorem 4.6 *Let $Y_n(\xi), Y_m(\xi)$ be two spherical harmonics. Then*

$$\int_{S^{p-1}} Y_n(\xi) Y_m(\xi) d\Omega_{p-1} = 0, \quad \text{if } n \neq m.$$

That is, spherical harmonics of different degrees are orthogonal over the sphere.

Proof Let us perform the computation using the harmonic homogeneous polynomials H_n and H_m where $Y_n = H_n|_{S^{p-1}}$ and $Y_m = H_m|_{S^{p-1}}$.

Now, we start with the divergence theorem in p dimensions for a vector A ,

$$\int_{B^p} \nabla_p \cdot A(x) d^p x = \int_{S^{p-1}} A(\xi) \cdot \xi d\Omega_{p-1},$$

apply it twice for the vectors $H_n \nabla H_m$ and $H_m \nabla H_n$, and subtract the results¹:

$$\begin{aligned} & \int_{B^p} \nabla_p \cdot [H_n(x) \nabla_p H_m(x) - H_m(x) \nabla_p H_n(x)] d^p x \\ &= \int_{S^{p-1}} [H_n(\xi) \nabla_p H_m(\xi) - H_m(\xi) \nabla_p H_n(\xi)] \cdot \xi d\Omega_{p-1}, \end{aligned}$$

or

$$0 = \int_{S^{p-1}} [H_n(\xi) \nabla_p H_m(\xi) - H_m(\xi) \nabla_p H_n(\xi)] \cdot \xi d\Omega_{p-1}, \quad (4.9)$$

where we used the property

$$\nabla_p \cdot (H_n \nabla_p H_m) = \nabla_p H_n \cdot \nabla_p H_m + H_n \Delta_p H_m,$$

and the fact $\Delta_p H_m = \Delta_p H_n = 0$.

Euler's equation (4.1) for a homogeneous polynomial,

$$\sum_{j=1}^p \frac{\partial H_n(\xi)}{\partial \xi_j} \xi_j = n H_n(\xi),$$

can be written in the form

$$\nabla_p H_n(\xi) \cdot \xi = n H_n(\xi).$$

With the help of this result, equation (4.9) takes the form

$$(m - n) \int_{S^{p-1}} H_n(\xi) H_m(\xi) d\Omega_{p-1} = 0.$$

But the integral is carried out over S^{p-1} where $Y_n = H_n$ and $Y_m = H_m$. This equation is thus equivalent to

$$(m - n) \int_{S^{p-1}} Y_n(\xi) Y_m(\xi) d\Omega_{p-1} = 0.$$

By hypothesis, $n \neq m$; therefore Y_n, Y_m are orthogonal over the sphere. ■

¹Incidentally, we point out that the identity thus obtained for any two functions f, g

$$\int_{B^p} (f \Delta_p g - g \Delta_p f) d^p x = \int_{S^{p-1}} (f \nabla_p g - g \nabla_p f) \cdot \xi d\Omega_{p-1},$$

is known as *Green's theorem*. We may also observe that $\nabla_p f \cdot \xi$ is the directional derivative of f along ξ and write

$$\int_{B^p} (f \Delta_p g - g \Delta_p f) d^p x = \int_{S^{p-1}} \left(f \frac{\partial g}{\partial \xi} - g \frac{\partial f}{\partial \xi} \right) d\Omega_{p-1}.$$

Given a set of $N(p, n)$ linearly independent spherical harmonics of degree n , we can use the Gram-Schmidt orthonormalization procedure to produce an orthonormal set of spherical harmonics, i.e., a set

$$\{Y_{n,i}(\xi)\}_{i=1}^{N(p,n)} \quad \text{with} \quad \int_{S^{p-1}} Y_{n,i}(\xi) Y_{n,j}(\xi) d\Omega_{p-1} = \delta_{ij}, \quad (4.10)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

is the Kronecker delta. For the remainder of this chapter, unless indicated otherwise, we will let $Y_{n,i}(\xi)$ denote an n -th degree spherical harmonic belonging to an orthonormal set of $N(p, n)$ such functions, as in (4.10).

In what follows, let R be an orthogonal matrix that acts on ξ as a rotation of coordinates. Notice that since the integration is taken over the entire sphere in (4.10), the orthonormal set of spherical harmonics remains orthonormal in a rotated coordinate frame. That is,

$$\int_{S^{p-1}} Y_{n,i}(R\xi) Y_{n,j}(R\xi) d\Omega_{p-1} = \delta_{ij}. \quad (4.11)$$

Proposition 4.7 *If $Y_n(\xi)$ is a spherical harmonic of degree n , then $Y'_n(\xi) = Y_n(R\xi)$ is also a spherical harmonic of degree n , for any rotation matrix R .*

Proof Let $Y_n(\xi)$ be a spherical harmonic of degree n . Then there exists a harmonic homogeneous polynomial $H_n(x)$ of degree n such that $Y_n = H_n|_{S^{p-1}}$. Denote $H'_n(x) = H_n(Rx)$. We claim that $Y'_n = H'_n|_{S^{p-1}}$. To see this, first notice that $H'_n(x)$ is a polynomial in x_1, x_2, \dots, x_p . Indeed, $H'_n(x) = H_n(Rx)$ is a linear combination of powers of the $\sum_{j=1}^p R_{ij}x_j$ and thus a linear combination of powers of the x_j . Next, notice that H'_n is homogeneous of degree n ,

$$H'_n(tx) = H_n(tRx) = t^n H_n(Rx) = t^n H'_n(x).$$

Finally, notice that H'_n is harmonic, by Proposition 2.1. Restricting H'_n to the unit sphere thus gives a spherical harmonic of degree n , $Y'_n(\xi)$. ■

Since the set $\{Y_{n,i}(\xi)\}_{i=1}^{N(p,n)}$ in (4.10) is a maximal linearly independent set of spherical harmonics of degree n , it serves as a basis for all such functions. We have just shown that $Y_{n,j}(R\xi)$ is a spherical harmonic of degree n provided $Y_{n,j}(\xi)$ is as well. Thus, we can write $Y_{n,j}(R\xi)$ in terms of the basis functions:

$$Y_{n,j}(R\xi) = \sum_{\ell=1}^{N(p,n)} C_{\ell j} Y_{n,\ell}(\xi).$$

Using this expression to rewrite the integral in (4.11), we can show that the matrix C defined in the above expression is orthogonal. Indeed,

$$\begin{aligned}
 \delta_{ij} &= \int_{\xi \in S^{p-1}} \left(\sum_{k=1}^{N(p,n)} C_{ki} Y_{n,k}(\xi) \right) \left(\sum_{\ell=1}^{N(p,n)} C_{\ell j} Y_{n,\ell}(\xi) \right) d\Omega_{p-1} \\
 &= \sum_{k,\ell=1}^{N(p,n)} C_{ki} C_{\ell j} \int_{\xi \in S^{p-1}} Y_{n,k}(\xi) Y_{n,\ell}(\xi) d\Omega_{p-1} \\
 &= \sum_{k,\ell=1}^{N(p,n)} C_{ki} C_{\ell j} \delta_{k\ell} = \sum_{k=1}^{N(p,n)} C_{ik}^t C_{kj}.
 \end{aligned}$$

For the following discussion, let ξ, η be two unit vectors. Let us consider the function given by

$$F_n(\xi, \eta) = \sum_{j=1}^{N(p,n)} Y_{n,j}(\xi) Y_{n,j}(\eta). \quad (4.12)$$

Lemma 4.8 *The function F_n defined above is invariant under a rotation of coordinates.*

Proof Let R be a rotation matrix. Then, using the orthogonal matrix C discussed above,

$$\begin{aligned}
 F_n(R\xi, R\eta) &= \sum_{j=1}^{N(p,n)} Y_{n,j}(R\xi) Y_{n,j}(R\eta) \\
 &= \sum_{j=1}^{N(p,n)} \left(\sum_{\ell=1}^{N(p,n)} C_{\ell j} Y_{n,\ell}(\xi) \right) \left(\sum_{m=1}^{N(p,n)} C_{mj} Y_{n,m}(\eta) \right)
 \end{aligned}$$

so

$$\begin{aligned}
 F_n(R\xi, R\eta) &= \sum_{\ell,m=1}^{N(p,n)} Y_{n,\ell}(\xi) Y_{n,m}(\eta) \left(\sum_{j=1}^{N(p,n)} C_{\ell j} C_{jm}^t \right) \\
 &= \sum_{\ell=1}^{N(p,n)} Y_{n,\ell}(\xi) Y_{n,\ell}(\eta) = F_n(\xi, \eta),
 \end{aligned}$$

as claimed. ■

Since the dot product $\langle \xi, \eta \rangle$ is also invariant under the rotation R , this suggests that $F_n(\xi, \eta)$ could be a function of $\langle \xi, \eta \rangle$ alone. In fact, this is the case, as shown in the following lemma.

Lemma 4.9 *If F_n is defined as in (4.12), then $F_n(\xi, \eta) = p(\langle \xi, \eta \rangle)$ where $p(t)$ is a polynomial.*

Proof A rotation of coordinates leaves $\langle \xi, \eta \rangle$ invariant and, by the above lemma, does not change $F_n(\xi, \eta)$ either. For some $-1 \leq t \leq 1$, there exists a rotation R that sends

$$\begin{aligned}\xi &\xrightarrow{R} \xi' = (t, \sqrt{1-t^2}, 0, \dots, 0), \\ \eta &\xrightarrow{R} \eta' = (1, 0, \dots, 0).\end{aligned}$$

To see this, just rotate coordinates so that η points along the x_1 -axis. Then rotate coordinates around the x_1 -axis until the component of ξ orthogonal to η points along the x_2 -axis.

Notice that $\langle \xi, \eta \rangle = \langle \xi', \eta' \rangle = t$. Since $F_n(\xi, \eta)$ is a sum of products of spherical harmonics, which are each polynomials in the components of ξ or η , F_n is a polynomial, say p , in the components of its arguments, i.e.,

$$F_n(\xi, \eta) = F_n(\xi', \eta') = p(t, \sqrt{1-t^2}). \quad (4.13)$$

We can impose another rotation of coordinates, again without changing F_n or $\langle \xi, \eta \rangle$. Let \tilde{R} be the transformation that rotates vectors by π radians about the x_1 -axis, i.e.,

$$\begin{aligned}\xi' &\xrightarrow{\tilde{R}} \xi'' = (t, -\sqrt{1-t^2}, 0, \dots, 0), \\ \eta' &\xrightarrow{\tilde{R}} \eta'' = (1, 0, \dots, 0).\end{aligned}$$

Just as in (4.13), we conclude that

$$F_n(\xi, \eta) = F_n(\xi'', \eta'') = p(t, -\sqrt{1-t^2}). \quad (4.14)$$

From (4.13) and (4.14), we see that $\sqrt{1-t^2}$ must appear only with even powers in F_n . Thus, F_n is really a polynomial in t and $1-t^2$, which is just a polynomial in t . Since $t = \langle \xi, \eta \rangle$, the lemma is proved. \blacksquare

4.3 Legendre Polynomials

Theorem 4.10 *Let $\eta = (1, 0, \dots, 0)$ and let $L_n(x)$ be a harmonic homogeneous polynomial of degree n satisfying*

$$(i) \quad L_n(\eta) = 1,$$

$$(ii) \quad L_n(Rx) = L_n(x) \text{ for all rotation matrices } R \text{ such that } R\eta = \eta.$$

Then $L_n(x)$ is the only harmonic homogeneous polynomial of degree n obeying these properties. In particular, these two properties uniquely determine the corresponding spherical harmonic $L_n|_{S^{p-1}}$. Moreover, this spherical harmonic $L_n(\xi)$ is a polynomial in $\langle \xi, \eta \rangle$.

Proof Let $\xi \in S^{p-1}$. Then there exist $\nu \in S^{p-1}$ such that $\langle \nu, \eta \rangle = 0$ and $t \in [-1, 1]$ such that $\xi = t\eta + \sqrt{1-t^2}\nu$. Notice $\langle \xi, \eta \rangle = t$ and that

$$\xi_1 = t, \quad \xi_2^2 + \cdots + \xi_p^2 = 1 - t^2. \quad (4.15)$$

As in the proof of Theorem 4.4, we can write

$$L_n(x) = \sum_{j=1}^n x_1^j h_{n-j}(x_2, x_3, \dots, x_p), \quad (4.16)$$

where the h_{n-j} are homogeneous polynomials of degree $n-j$. Let R be a rotation matrix as described in property (ii) in the statement of the theorem. When R acts on x , it sends

$$x_1, x_2, \dots, x_p \xrightarrow{R} x_1, x'_2, \dots, x'_p.$$

By property (ii),

$$0 = L_n(x) - L_n(Rx) = \sum_{j=1}^n x_1^j [h_{n-j}(x_2, \dots, x_p) - h_{n-j}(x'_2, \dots, x'_p)],$$

and using the linear independence of the x_1^j , we see that all the $h_{n-j}(x_2, \dots, x_p)$ are invariant under the rotation of coordinates R . Thus, these polynomials must depend only on the radius $\sqrt{x_2^2 + \cdots + x_p^2}$, i.e.,

$$h_{n-j}(x_2, \dots, x_p) = c_{n-j} \left(\sqrt{x_2^2 + \cdots + x_p^2} \right)^{n-j}, \quad (4.17)$$

where the c_{n-j} are constants, and $c_{n-j} = 0$ for all odd $n-j$ since the h_{n-j} must be polynomials. Property (i) gives us one of these coefficients,

$$1 = L_n(\eta) = c_0$$

since all the x_2, \dots, x_p are zero for this vector. Since the form of the h_{n-j} is given by (4.17), knowing c_0 is enough to determine the rest of the h_{n-j} using the recursive relation (4.6) and the fact that the c_{n-j} are zero for odd $n-j$. Therefore, L_n as well as the spherical harmonic $L_n|_{S^{p-1}}$ are uniquely determined by properties (i) and (ii). Finally, using (4.16), (4.17), and (4.15),

$$L_n(\xi) = \sum_{\substack{j=0 \\ n-j=\text{even}}}^n \xi_1^j c_{n-j} (\xi_2^2 + \cdots + \xi_p^2)^{(n-j)/2},$$

or

$$L_n(\xi(t)) = \sum_{\substack{j=0 \\ n-j=\text{even}}}^n t^j c_{n-j} (1-t^2)^{(n-j)/2}. \quad (4.18)$$

That is, $L_n(\xi)$ is a polynomial in $t = \langle \xi, \eta \rangle$, and the theorem is proved. \blacksquare

Definition We call the polynomial $L_n(\xi)$ introduced in Theorem 4.10 the *Legendre polynomial*² of degree n . Written in terms of the variable t , we denote it by $P_n(t)$.

Remark We can quickly obtain some properties of Legendre polynomials. First, from (4.18), we can see $P_n(t)$ is a polynomial of degree n . We can also compute

$$1 = L_n(\eta) = P_n(\langle \eta, \eta \rangle) = P_n(1).$$

We can determine the parity of the Legendre polynomials in more than one way. First, from (4.18), we can see that if n is even, P_n contains only even powers of t , while if n is odd, P_n contains only odd powers of t . Alternatively, we can see that

$$\begin{aligned} P_n(-t) &= P_n(\langle -\xi, \eta \rangle) = L_n(-\xi) = (-1)^n L_n(\xi) \\ &= (-1)^n P_n(\langle \xi, \eta \rangle) = (-1)^n P_n(t). \end{aligned} \quad (4.19)$$

Either way, we determine that P_n is even whenever n is even, and P_n is odd whenever n is odd.

In the following theorem, we demonstrate how to write the Legendre polynomials in terms of an orthonormal set of spherical harmonics.

Theorem 4.11 (Addition Theorem for Legendre Polynomials) *Let $\{Y_{n,j}(\xi)\}_{j=1}^{N(p,n)}$ be an orthonormal set of n -th degree spherical harmonics. Then the Legendre polynomial of degree n may be written as*

$$P_n(\langle \xi, \eta \rangle) = \frac{\Omega_{p-1}}{N(p,n)} \sum_{j=1}^{N(p,n)} Y_{n,j}(\xi) Y_{n,j}(\eta). \quad (4.20)$$

²We follow the naming convention used in [5] and common in physics. Mathematicians usually refer to these as Legendre polynomials only when $p = 3$ and as *ultraspherical polynomials* for arbitrary p .

Proof Since (4.20) is invariant under coordinate rotations, we can choose $\eta = (1, 0, \dots, 0)$. Consider again the function $F_n(\xi, \eta)$ defined in (4.12). Since we have already defined the vector η , we will think of it as a fixed parameter in the function F_n . For now, we will write $F_n(\xi; \eta)$ and think of F_n as a function of one vector ξ .

First, notice that F_n , being a linear combination of spherical harmonics $Y_{n,j}(\xi)$ (since we consider the $Y_{n,j}(\eta)$ to be constants), is itself a spherical harmonic, i.e., the restriction of a harmonic homogeneous polynomial to the unit sphere. Next, notice that any rotation of coordinates R that leaves η fixed leaves the function F_n invariant. Indeed, using Lemma 4.8 or Lemma 4.9,

$$F(R\xi; \eta) = F(R\xi; R\eta) = F(\xi; \eta) \text{ for all } R \text{ such that } R\eta = \eta.$$

Notice that in the application of the rotation R , we only rotate the coordinates of ξ in F_n since we consider η a fixed parameter, though this made no difference in the calculation.

Let us normalize the function F_n by dividing it by the constant $F_n(\eta; \eta)$. Thus, we have found a function $F_n(\xi; \eta)/F_n(\eta; \eta)$ that obeys all the properties of the Legendre polynomial described in the statement of Theorem 4.10. Since, by the same theorem, these properties uniquely define the Legendre polynomial, we conclude

$$P_n(\langle \xi, \eta \rangle) = \frac{F_n(\xi, \eta)}{F_n(\eta, \eta)}.$$

To complete the proof, we will compute $F_n(\eta, \eta)$. Since, by Lemma 4.9, $F_n(\eta, \eta)$ depends only on the inner product $\langle \eta, \eta \rangle = 1$, it is a constant. Thus,

$$\begin{aligned} F_n(\eta, \eta)\Omega_{p-1} &= \int_{\eta \in S^{p-1}} F_n(\eta, \eta) d\Omega_{p-1} \\ &= \int_{\eta \in S^{p-1}} \sum_{j=1}^{N(p,n)} Y_{n,j}(\eta)^2 d\Omega_{p-1} \\ &= \sum_{j=1}^{N(p,n)} \int_{\eta \in S^{p-1}} Y_{n,j}(\eta)^2 d\Omega_{p-1} = N(p, n), \end{aligned}$$

using the orthonormality of the n -th degree spherical harmonics. We see that

$$P_n(\langle \xi, \eta \rangle) = \frac{F_n(\xi, \eta)}{F_n(\eta, \eta)} = \frac{\Omega_{p-1}}{N(p, n)} \sum_{j=1}^{N(p,n)} Y_{n,j}(\xi) Y_{n,j}(\eta),$$

and we are finished. ■

In addition, we can expand spherical harmonics in terms of Legendre polynomials. To show this, we will need first the following result.

Lemma 4.12 *For any set $\{Y_{n,j}\}_{j=1}^k$ of $k \leq N(p, n)$ linearly independent n -th degree spherical harmonics, there exists a set $\{\eta_i\}_{i=1}^k$ of unit vectors such that the $k \times k$ determinant*

$$\begin{vmatrix} Y_{n,1}(\eta_1) & Y_{n,1}(\eta_2) & \cdots & Y_{n,1}(\eta_k) \\ Y_{n,2}(\eta_1) & Y_{n,2}(\eta_2) & \cdots & Y_{n,2}(\eta_k) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n,k}(\eta_1) & Y_{n,k}(\eta_2) & \cdots & Y_{n,k}(\eta_k) \end{vmatrix} \quad (4.21)$$

is nonzero.

Proof We will prove the lemma by induction on k . First, consider a linearly independent set $\{Y_{n,1}\}$ of one spherical harmonic of degree n . $Y_{n,1}$ cannot be the zero function. Then, there exists a unit vector, call it η_1 , such that the determinant $|Y_{n,1}(\eta_1)| = Y_{n,1}(\eta_1) \neq 0$. Thus, the lemma holds for the case $k = 1$. Now, suppose the lemma holds for some $k = \ell - 1 \leq N(p, n) - 1$, and let $\{Y_{n,j}\}_{j=1}^\ell$ be a set of $\ell \leq N(p, n)$ linearly independent n -th degree spherical harmonics. By the induction hypothesis, we can choose a set $\{\eta_i\}_{i=1}^{\ell-1}$ of unit vectors such that the $(\ell - 1) \times (\ell - 1)$ determinant

$$\Delta_\ell = \begin{vmatrix} Y_{n,1}(\eta_1) & Y_{n,1}(\eta_2) & \cdots & Y_{n,1}(\eta_{\ell-1}) \\ Y_{n,2}(\eta_1) & Y_{n,2}(\eta_2) & \cdots & Y_{n,2}(\eta_{\ell-1}) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n,\ell-1}(\eta_1) & Y_{n,\ell-1}(\eta_2) & \cdots & Y_{n,\ell-1}(\eta_{\ell-1}) \end{vmatrix} \neq 0. \quad (4.22)$$

Now consider the spherical harmonic defined by the $\ell \times \ell$ determinant,

$$\Delta = \begin{vmatrix} Y_{n,1}(\eta_1) & Y_{n,1}(\eta_2) & \cdots & Y_{n,1}(\eta_{\ell-1}) & Y_{n,1}(\xi) \\ Y_{n,2}(\eta_1) & Y_{n,2}(\eta_2) & \cdots & Y_{n,2}(\eta_{\ell-1}) & Y_{n,2}(\xi) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{n,\ell-1}(\eta_1) & Y_{n,\ell-1}(\eta_2) & \cdots & Y_{n,\ell-1}(\eta_{\ell-1}) & Y_{n,\ell-1}(\xi) \\ Y_{n,\ell}(\eta_1) & Y_{n,\ell}(\eta_2) & \cdots & Y_{n,\ell}(\eta_{\ell-1}) & Y_{n,\ell}(\xi) \end{vmatrix}. \quad (4.23)$$

If we compute this determinant by performing a cofactor expansion down the last column and indicating by Δ_j the minor determinant corresponding to $Y_{n,j}(\xi)$,

$$\Delta = \sum_{j=1}^{\ell} (-1)^{\ell+j} \Delta_j Y_{n,j}(\xi),$$

we can see that we will have a linear combination of spherical harmonics $Y_{n,j}(\xi)$ which are linearly independent. Thus, for the determinant to vanish identically, all the coefficients of the $Y_{n,j}(\xi)$ must vanish. But notice that the coefficient of $Y_{n,\ell}(\xi)$ is the determinant (4.22) which does not vanish. Thus, the spherical harmonic given by the determinant (4.23) is not the zero function; i.e., there exists a unit vector $\xi = \eta_\ell$ such that the determinant (4.23) is nonzero. Therefore, the lemma holds for the case $k = \ell$ and, by induction, for all $k \leq N(p, n)$. ■

Theorem 4.13 *For any spherical harmonic $Y_n(\xi)$ of degree n , there exist coefficients a_k and unit vectors η_k such that*

$$Y_n(\xi) = \sum_{k=1}^{N(p,n)} a_k P_n(\langle \xi, \eta_k \rangle).$$

Proof Let $\{Y_{n,j}(\xi)\}_{j=1}^{N(p,n)}$ be an orthonormal set of n -th degree spherical harmonics, and let $Y_n(\xi)$ be any spherical harmonic of degree n . For a unit vector η , we can write

$$P_n(\langle \xi, \eta \rangle) = \frac{\Omega_{p-1}}{N(p, n)} \sum_{j=1}^{N(p,n)} Y_{n,j}(\xi) Y_{n,j}(\eta), \quad (4.24)$$

by Theorem 4.11. Let us choose a set of unit vectors η_j such that the determinant (4.21) with $k = N(p, n)$ is nonzero; this is possible by Lemma 4.12. Replacing η by η_j in (4.24) for $1 \leq j \leq N(p, n)$ creates the system of equations

$$\frac{N(p, n)}{\Omega_{p-1}} \begin{pmatrix} P_n(\langle \xi, \eta_1 \rangle) \\ P_n(\langle \xi, \eta_2 \rangle) \\ \vdots \\ P_n(\langle \xi, \eta_{N(p,n)} \rangle) \end{pmatrix} = \begin{pmatrix} Y_{n,1}(\eta_1) & \cdots & Y_{n,N(p,n)}(\eta_1) \\ Y_{n,1}(\eta_2) & \cdots & Y_{n,N(p,n)}(\eta_2) \\ \vdots & \ddots & \vdots \\ Y_{n,1}(\eta_{N(p,n)}) & \cdots & Y_{n,N(p,n)}(\eta_{N(p,n)}) \end{pmatrix} \begin{pmatrix} Y_{n,1}(\xi) \\ Y_{n,2}(\xi) \\ \vdots \\ Y_{n,N(p,n)}(\xi) \end{pmatrix}.$$

The determinant of the $N(p, n) \times N(p, n)$ coefficient matrix on the right-hand side of this system has nonzero determinant by our choice of the vectors η_j . Thus, this system is invertible; i.e., there exist coefficients $c_{\ell,m}$ such that

$$Y_{n,\ell}(\xi) = \sum_{m=1}^{N(p,n)} c_{\ell,m} P_n(\langle \xi, \eta_m \rangle), \text{ for each } 1 \leq \ell \leq N(p, n). \quad (4.25)$$

Now since any spherical harmonic, and in particular $Y_n(\xi)$, can be expanded in terms of the basis functions $\{Y_{n,i}(\xi)\}_{i=1}^{N(p,n)}$, there exist coefficients a_k such that

$$Y_n(\xi) = \sum_{k=1}^{N(p,n)} a_k P_n(\langle \xi, \eta_k \rangle),$$

by (4.25), and the theorem is proved. \blacksquare

We will now list and prove several basic properties of spherical harmonics and Legendre polynomials, some of which are useful for making estimates. The proofs are straightforward.

Lemma 4.14 *For any spherical harmonic $Y_n(\xi)$,*

$$Y_n(\xi) = \frac{N(p, n)}{\Omega_{p-1}} \int_{\eta \in S^{p-1}} Y_n(\eta) P_n(\langle \xi, \eta \rangle) d\Omega_{p-1}. \quad (4.26)$$

Proof Let $Y_n(\xi)$ be any n -th degree spherical harmonic, and let $\{Y_{n,j}\}_{j=1}^{N(p,n)}$ be an orthonormal set of such functions. Then, we can expand $Y_n(\eta)$ in terms of this basis; i.e., for some coefficients a_j ,

$$Y_n(\eta) = \sum_{j=1}^{N(p,n)} a_j Y_{n,j}(\eta).$$

Using this expansion and Theorem 4.11 to rewrite $P_n(\langle \xi, \eta \rangle)$, the right-hand side of (4.26) becomes

$$\frac{N(p, n)}{\Omega_{p-1}} \int_{\eta \in S^{p-1}} \left[\sum_{j=1}^{N(p,n)} a_j Y_{n,j}(\eta) \right] \left[\frac{\Omega_{p-1}}{N(p, n)} \sum_{k=1}^{N(p,n)} Y_{n,k}(\xi) Y_{n,k}(\eta) \right] d\Omega_{p-1},$$

or

$$\sum_{j,k=1}^{N(p,n)} a_j Y_{n,k}(\xi) \left[\int_{\eta \in S^{p-1}} Y_{n,j}(\eta) Y_{n,k}(\eta) d\Omega_{p-1} \right] = \sum_{j=1}^{N(p,n)} a_j Y_{n,j}(\xi) = Y_n(\xi),$$

as required. \blacksquare

Proposition 4.15 *The Legendre polynomials $P_n(t)$ are bounded*

$$|P_n(t)| \leq 1, \text{ for all } t \in [0, 1] \quad (4.27)$$

and obey the following normalization condition

$$\int_{\xi \in S^{p-1}} P_n(\langle \xi, \eta \rangle)^2 d\Omega_{p-1} = \frac{\Omega_{p-1}}{N(p, n)}. \quad (4.28)$$

Proof We rewrite $P_n(t)^2$ using Theorem 4.11. Then, we use the Cauchy-Schwarz inequality, viewing the sum of products in this expression as a dot product, to derive the required result:

$$\begin{aligned} P_n(\langle \xi, \eta \rangle)^2 &= \left[\frac{\Omega_{p-1}}{N(p, n)} \sum_{j=1}^{N(p, n)} Y_{n,j}(\xi) Y_{n,j}(\eta) \right]^2 \\ &\leq \left[\frac{\Omega_{p-1}}{N(p, n)} \sum_{j=1}^{N(p, n)} Y_{n,j}(\xi)^2 \right] \left[\frac{\Omega_{p-1}}{N(p, n)} \sum_{j=1}^{N(p, n)} Y_{n,j}(\eta)^2 \right], \end{aligned}$$

so

$$P_n(\langle \xi, \eta \rangle)^2 \leq P_n(\langle \xi, \xi \rangle) P_n(\langle \eta, \eta \rangle) = P_n(1)^2 = 1,$$

proving the first result.

Again, we use Theorem 4.11 to rewrite the integral on the left side of (4.28) as

$$\int_{\xi \in S^{p-1}} \frac{\Omega_{p-1}^2}{N(p, n)^2} \sum_{j,k=1}^{N(p, n)} Y_{n,j}(\xi) Y_{n,j}(\eta) Y_{n,k}(\xi) Y_{n,k}(\eta) d\Omega_{p-1}$$

which then becomes

$$\frac{\Omega_{p-1}^2}{N(p, n)^2} \sum_{j=1}^{N(p, n)} Y_{n,j}(\eta) Y_{n,j}(\eta) = \frac{\Omega_{p-1}}{N(p, n)} P_n(\langle \eta, \eta \rangle) = \frac{\Omega_{p-1}}{N(p, n)},$$

thus proving the second result. ■

Proposition 4.16 *The spherical harmonics $Y_n(\xi)$ satisfy the following inequality*

$$|Y_n(\xi)| \leq \sqrt{\frac{N(p, n)}{\Omega_{p-1}} \int_{\eta \in S^{p-1}} Y_n(\eta)^2 d\Omega_{p-1}}. \quad (4.29)$$

Proof We start by taking the square of equation (4.26):

$$Y_n(\xi)^2 = \frac{N(n, p)^2}{\Omega_{p-1}^2} \left[\int_{\eta \in S^{p-1}} Y_n(\eta) P_n(\langle \xi, \eta \rangle) d\Omega_{p-1} \right]^2.$$

Viewing the integral as an inner product, we apply the Cauchy-Schwarz inequality,

$$Y_n(\xi)^2 \leq \frac{N(n, p)^2}{\Omega_{p-1}^2} \left[\int_{\eta \in S^{p-1}} Y_n(\eta)^2 d\Omega_{p-1} \right] \left[\int_{\eta \in S^{p-1}} P_n(\langle \xi, \eta \rangle)^2 d\Omega_{p-1} \right].$$

Thus,

$$Y_n(\xi)^2 \leq \frac{N(p, n)}{\Omega_{p-1}} \int_{S^{p-1}} Y_n(\eta)^2 d\Omega_{p-1},$$

where we used property (4.28) in the last step. ■

We will now begin to investigate the properties of Legendre polynomials as orthogonal polynomials. Let us rewrite the integral in (4.28). Note that the integrand depends only on the inner product $\langle \xi, \eta \rangle$ and that we integrate ξ over the surface of the $(p-1)$ -sphere. We can take advantage of these observations to reduce the $(p-1)$ -dimensional integral to a one-dimensional integral.

Since we integrate over the entire sphere, we can perform any rotation of coordinates without changing the value of the integral. Let us impose a coordinate change R that aligns the unit vector η along the x_p -axis, which we will picture pointing “north,” i.e., take $\eta = (0, \dots, 0, 1)$. If we let $t = \langle \xi, \eta \rangle$, we can write the unit vector ξ as

$$\xi = \langle \xi, \eta \rangle \eta + (\xi - \langle \xi, \eta \rangle \eta) = t\eta + \sqrt{1-t^2} \nu,$$

for some unit vector ν normal to η . Notice that any such ν gives the same value for the inner product $\langle \xi, \eta \rangle$ and thus the same value for the integrand in (4.28). Note further that the collection of all such vectors ν ,

$$\{\nu \in \mathbb{R}^p : |\nu| = 1, \langle \nu, \eta \rangle = 0\},$$

forms a parallel of the $(p-1)$ -sphere which is a $(p-2)$ -sphere:

$$\{\nu \in \mathbb{R}^p : |\nu| = 1, \nu_p = 0\} = S^{p-2}.$$

A visual representation is depicted in Figure 4.1.

To get some intuition, let's think of the familiar 3-dimensional case. The 2-dimensional sphere can be thought as a sum of infinitesimal rings oriented along the parallels of the sphere. The infinitesimal ring defined by the azimuthal angle θ has an infinitesimal thickness $d\theta$ and, therefore, it corresponds to a solid angle

$$d\Omega_2 = (2\pi \sin \theta) d\theta.$$

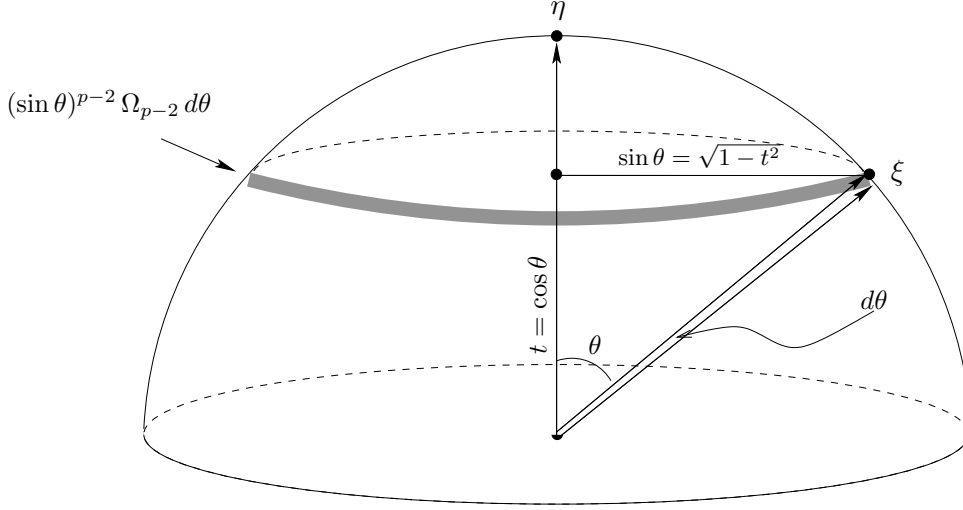


Figure 4.1: In the reduction of a spherically symmetric $(p-1)$ -dimensional integral to a 1-dimensional integral, we imagine the $(p-1)$ -dimensional sphere as a sum of infinitesimal $(p-2)$ -dimensional spheres (parallels) of thickness $d\theta$.

The term inside the parenthesis is the length of the ring; multiplied by its thickness it gives the “area” of the ring. We can rewrite $d\Omega_2$ as $-2\pi d(\cos \theta)$ and thus

$$d\Omega_2 = -\Omega_1 dt$$

in terms of the variable t .

Similarly, in p dimensions

$$d\Omega_{p-1} = \Omega_{p-2} (\sin \theta)^{p-2} d\theta .$$

The factor $(\sin \theta)^{p-2}$ is easy to explain: The radius of the $(p-2)$ -sphere is $R = \sin \theta$; its volume will be proportional to R^{p-2} . It is straightforward to express $d\Omega_{p-1}$ in terms of t :

$$\begin{aligned} d\Omega_{p-1} &= -\Omega_{p-2} (\sin \theta)^{p-3} d(\cos \theta) \\ &= -\Omega_{p-2} (1 - t^2)^{\frac{p-3}{2}} dt. \end{aligned}$$

Now we are ready to write (4.28) as a 1-dimensional integral over t . Indeed,

$$\int_{\xi \in S^{p-1}} P_n(\langle \xi, \eta \rangle)^2 d\Omega_{p-1} = \int_{-1}^1 P_n(t)^2 (1 - t^2)^{\frac{p-3}{2}} \Omega_{p-2} dt,$$

which implies

$$\int_{-1}^1 P_n(t)^2 (1-t^2)^{\frac{p-3}{2}} dt = \frac{\Omega_{p-1}}{N(p, n)\Omega_{p-2}}.$$

This expression gives us the norm of the n -th Legendre polynomial with respect to the weight $w(t) = (1-t^2)^{\frac{p-3}{2}}$, namely

$$\|P_n(t)\|_w = \sqrt{\langle P_n(t), P_n(t) \rangle_w} = \sqrt{\frac{\Omega_{p-1}}{N(p, n)\Omega_{p-2}}}. \quad (4.30)$$

Note that in coming up with this fact, we have essentially proved the following lemma, which will be used to show the next few results.

Lemma 4.17 *Let η be a unit vector and f be a function. Then*

$$\int_{\xi \in S^{p-1}} f(\langle \xi, \eta \rangle) d\Omega_{p-1} = \Omega_{p-2} \int_{-1}^1 f(t)(1-t^2)^{(p-3)/2} dt.$$

Theorem 4.18 *Any two distinct Legendre polynomials $P_n(t), P_m(t)$ are orthogonal over the interval $[-1, 1]$ with respect to the weight $(1-t^2)^{\frac{p-3}{2}}$. That is,*

$$\int_{-1}^1 P_n(t)P_m(t)(1-t^2)^{\frac{p-3}{2}} dt = 0, \quad \text{for } n \neq m.$$

Proof Let $\eta = (1, 0, \dots, 0)$. The Legendre polynomial $P_n(\langle \xi, \eta \rangle)$ is equal to the spherical harmonic $L_n(\xi)$ having the properties listed in Theorem 4.10. By Theorem 4.6,

$$0 = \int_{\xi \in S^{p-1}} L_n(\xi)L_m(\xi) d\Omega_{p-1} = \int_{\xi \in S^{p-1}} P_n(\langle \xi, \eta \rangle)P_m(\langle \xi, \eta \rangle) d\Omega_{p-1}, \quad \text{for } n \neq m.$$

Since the integral on the right-hand side of the above equation depends only on the inner product $\langle \xi, \eta \rangle$, we can use Lemma 4.17 to rewrite this integral as

$$\Omega_{p-2} \int_{-1}^1 P_n(t)P_m(t)(1-t^2)^{\frac{p-3}{2}} dt = 0, \quad \text{for } n \neq m,$$

thus completing the proof. ■

The above theorem allows us to use the results from Chapter 3 to write down more properties of the Legendre polynomials. Using the comment just above Section 3.4, we can find the Rodrigues formula for the Legendre polynomials in p dimensions. Theorem 4.18 tells us that the Legendre polynomials are orthogonal with respect to $w(t) = (1 - t^2)^{(p-3)/2}$ so that

$$\begin{aligned} P_n(t) &= c_n(1 - t^2)^{-(p-3)/2} \left(\frac{d}{dt} \right)^n \left[(1 - t^2)^{(p-3)/2} (1 - t^2)^n \right] \\ &= c_n(1 - t^2)^{(3-p)/2} \left(\frac{d}{dt} \right)^n (1 - t^2)^{n+(p-3)/2}, \end{aligned}$$

and it remains to compute c_n . We know that $P_n(1) = 1$, so

$$1 = c_n(1 - t^2)^{(3-p)/2} \left(\frac{d}{dt} \right)^n (1 - t^2)^{n+(p-3)/2} \Big|_{t=1}.$$

Carrying out two differentiations,

$$\begin{aligned} 1 &= c_n(1 - t^2)^{(3-p)/2} \left(\frac{d}{dt} \right)^{n-1} \left(\frac{n + (p-3)}{2} \right) (-2t)(1 - t^2)^{n-1+(p-3)/2} \Big|_{t=1} \\ &= c_n(1 - t^2)^{(3-p)/2} \left(\frac{d}{dt} \right)^{n-2} \left[\left(\frac{n + (p-3)}{2} \right)_2 (-2t)^2 (1 - t^2)^{n-2+(p-3)/2} + \dots \right] \Big|_{t=1} \end{aligned}$$

where we have used the falling factorial notation and left terms that are higher order in $1 - t^2$ in the “...” because they will vanish when we substitute $t = 1$. Continuing the pattern,

$$\begin{aligned} 1 &= c_n(1 - t^2)^{(3-p)/2} \left[(n + (p-3)/2)_n (-2t)^n (1 - t^2)^{n-n+(p-3)/2} + \dots \right] \Big|_{t=1} \\ &= c_n (n + (p-3)/2)_n (-2)^n, \end{aligned}$$

implying that

$$c_n = \frac{(-1)^n}{2^n (n + (p-3)/2)_n}.$$

We have shown the following.

Proposition 4.19 (Rodrigues Formula for Legendre Polynomials)

$$P_n(t) = \frac{(-1)^n}{2^n (n + (p-3)/2)_n} (1 - t^2)^{(3-p)/2} \left(\frac{d}{dt} \right)^n (1 - t^2)^{n+(p-3)/2}. \quad (4.31)$$

The above Rodrigues formula allows us to prove the following two properties of the Legendre polynomials.

Proposition 4.20 *In p dimensions, the Legendre polynomial $P_n(t)$ of degree n satisfies the differential equation*

$$(1 - t^2)P_n''(t) + (1 - p)tP_n'(t) + n(n + p - 2)P_n(t) = 0. \quad (4.32)$$

In what follows, we will often write P_n and w instead of $P_n(t)$ and $w(t)$ for simplicity. A portion of the proof will be left as a straightforward computation for the reader.

Proof Consider the expression $\frac{d}{dt} [(1 - t^2)^{(p-1)/2} P_n']$. We can use the product rule to rewrite this as

$$\left(\frac{p-1}{2}\right) (1 - t^2)^{(p-3)/2} (-2t) P_n' + (1 - t^2)^{(p-1)/2} P_n'',$$

or,

$$w [(1 - p)tP_n' + (1 - t^2)P_n'']. \quad (4.33)$$

The term in brackets is a polynomial of degree n which can be written as a linear combination of the first n Legendre polynomials. Hence

$$\frac{d}{dt} [(1 - t^2)wP_n'] = w \sum_{j=0}^n c_j P_j. \quad (4.34)$$

Multiplying each side by P_k , where $0 \leq k \leq n$ and integrating over the interval $[-1, 1]$, we get

$$c_k \|P_k\|^2 = \int_{-1}^1 P_k \frac{d}{dt} [(1 - t^2)wP_n'] dt.$$

Integrating the right-hand side by parts now twice and noticing that the boundary terms vanish, we find

$$\begin{aligned} c_k \|P_k\|^2 &= \underbrace{P_k(1 - t^2)wP_n'}_0 \Big|_{-1}^1 - \int_{-1}^1 P_k'(1 - t^2)wP_n' dt \\ &= \underbrace{-P_k'(1 - t^2)wP_n'}_0 \Big|_{-1}^1 + \int_{-1}^1 \left\{ \frac{d}{dt} [P_k'(1 - t^2)w] \right\} P_n dt. \end{aligned}$$

The expression in curly brackets can be written as wq_k , where q_k is a polynomial of degree k . We can then use Proposition 3.3 for the two integrals to discover that $c_k = 0$ for all $k < n$. Returning to equation (4.34), this result implies that

$$\frac{d}{dt} [(1 - t^2)wP_n'] = wc_n P_n.$$

To determine c_n , we will compute the coefficient of the highest power of t on each side of the above equation, using *Newton's binomial expansion*. By keeping only the highest-order term in each binomial expansion, the reader can show that the left-hand side becomes

$$\frac{(-1)^{(p-1)/2}}{2^n} \frac{(2n+p-3)_n}{(n+(p-3)/2)_n} n(n+p-2)t^{n+p-3} + \dots,$$

while the right-hand side becomes

$$c_n \frac{(-1)^{(p-3)/2}}{2^n} \frac{(2n+p-3)_n}{(n+(p-3)/2)_n} t^{n+p-3} + \dots$$

Comparing the above equations, we conclude that $c_n = -n(n+p-2)$. Inserting this into (4.34) and using (4.33), we find

$$(1-t^2)P_n'' + (1-p)tP_n' + n(n+p-2)P_n = 0,$$

completing the proof. ■

Proposition 4.21 *The Legendre polynomials in p dimensions satisfy the recurrence relation*

$$(n+p-2)P_{n+1}(t) - (2n+p-2)tP_n(t) + nP_{n-1}(t) = 0.$$

Just as in the previous proof, we will leave part of the following proof to the reader.

Proof We know from Proposition 3.5 that a relation of the form

$$P_{n+1} - (A_nt + B_n)P_n + C_nP_{n-1} = 0 \tag{4.35}$$

exists. We can quickly determine B_n . Recall from (4.19) that P_n is even (respectively, odd) whenever n is even (respectively, odd). Rewriting the above equation as

$$P_{n+1} - A_ntP_n + C_nP_{n-1} = B_nP_n,$$

we have an odd polynomial equal to an even polynomial, unless both sides vanish. Since the first option is not possible, we conclude that both sides of the equation must cancel, which implies that $B_n = 0$. We will use the notation and results of Proposition 3.5 to compute A_n, C_n . Keeping only the highest-order terms in each binomial expansion in the Rodrigues formula (4.31) and carrying out the derivatives, it is straightforward to show that the leading coefficient of P_n is

$$k_n = \frac{(2n+p-3)_n}{2^n(n+(p-3)/2)_n},$$

from which we determine the leading coefficient of P_{n+1} ,

$$k_{n+1} = \frac{(2n+p-1)_{n+1}}{2^{n+1}(n+(p-1)/2)_{n+1}},$$

allowing us to compute

$$A_n = \frac{k_{n+1}}{k_n} = \frac{2n+p-2}{n+p-2}.$$

Now, using (4.30),

$$\begin{aligned} C_n &= \frac{A_n}{A_{n-1}} \frac{\|P_n\|^2}{\|P_{n-1}\|^2} \\ &= \frac{2n+p-2}{n+p-2} \frac{n+p-3}{2n+p-4} \frac{\Omega_{p-1}}{N(p,n)\Omega_{p-2}} \frac{N(p,n-1)\Omega_{p-2}}{\Omega_{p-1}}. \end{aligned}$$

Using Theorem 4.4, we can compute

$$C_n = \frac{n}{n+p-2}.$$

Inserting these results into (4.35) and multiplying by $n+p-2$, we find the required result. \blacksquare

It is also interesting to observe that once we find the Legendre polynomials in $p=2$ and $p=3$ dimensions, they are determined for all higher dimensions. In particular, the following theorem proves, more specifically, that the Legendre polynomials for even p can be found from the $p=2$ case and that those for odd p can be found from the $p=3$ case. For the next theorem, we will let $P_{n,p}(t)$ denote the n -th degree Legendre polynomial in p dimensions.

Theorem 4.22 *For all $j = 0, 1, \dots, n$,*

$$P_{n-j,2j+p}(t) = \frac{((p-1)/2)_j}{(-n)_j(n+p-2)_j} \left(\frac{d}{dt} \right)^j P_{n,p}(t).$$

Proof Differentiating (4.32) once with respect to t , we get

$$(1-t^2)P_{n,p}''' + (-1-p)tP_{n,p}'' + (n-1)(n+p-1)P_{n,p}' = 0,$$

but this is just (4.32) again with the following substitutions:

$$\begin{aligned} P_{n,p} &\longmapsto P_{n,p}', \\ p &\longmapsto p+2, \\ n &\longmapsto n-1. \end{aligned}$$

Thus, solving this new differential equation for $P'_{n,p}$ will give

$$P'_{n,p}(t) \propto P_{n-1,p+2}(t).$$

Continuing in this way, if we differentiate (4.32) j times we see that

$$\left(\frac{d}{dt}\right)^j P_{n,p}(t) \propto P_{n-j,p+2j}(t)$$

for any $0 \leq j \leq n$. Since $P_{n-j,p+2j}(1) = 1$, in order to create an equality we must divide the left side of the above equation by its value at $t = 1$, i.e.,

$$P_{n-j,p+2j}(t) = \frac{\left(\frac{d}{dt}\right)^j P_{n,p}(t)}{\left(\frac{d}{dt}\right)^j P_{n,p}(t)|_{t=1}}.$$

Using the Rodrigues formula (4.31),

$$\left(\frac{d}{dt}\right)^j P_{n,p}(t)|_{t=1} = \frac{(-n)_j(n+p-2)_j}{((p-1)/2)_j},$$

as the reader can verify by computation. Therefore,

$$P_{n-j,2j+p}(t) = \frac{((p-1)/2)_j}{(-n)_j(n+p-2)_j} \left(\frac{d}{dt}\right)^j P_{n,p}(t),$$

completing the proof. ■

The following lemma will be useful in the proof of the theorem that follows immediately afterwards.

Lemma 4.23 *Let ξ and ζ be unit vectors, f a function, and $F(\zeta, \xi)$ given by*

$$F(\zeta, \xi) = \int_{\eta \in S^{p-1}} f(\langle \xi, \eta \rangle) P_n(\langle \eta, \zeta \rangle) d\Omega_{p-1}.$$

Then,

$$F(\zeta, \xi) = \Omega_{p-2} P_n(\langle \xi, \zeta \rangle) \int_{-1}^1 f(t) P_n(t) (1-t^2)^{(p-3)/2} dt.$$

Proof First, notice that F is invariant under any coordinate rotation R , i.e., $F(R\zeta, R\xi) = F(\zeta, \xi)$. Indeed, the rotation

$$\begin{aligned} \xi &\xrightarrow{R} \xi' = R\xi, \\ \zeta &\xrightarrow{R} \zeta' = R\zeta, \end{aligned}$$

can be undone by a change of variables

$$\eta \xrightarrow{R} \eta' = R\eta,$$

in the integration. Thus, we are allowed to choose our coordinates such that

$$\xi = (1, 0, \dots, 0), \quad \zeta = (s, \sqrt{1-s^2}, 0, \dots, 0),$$

so that $\langle \zeta, \xi \rangle = s$.

Let us think of F as a function of ζ alone and ξ as a fixed parameter; we will write $F(\zeta; \xi)$. The argument of F , namely ζ , only shows up inside the Legendre polynomial $P_n(\langle \eta, \zeta \rangle)$, which is a spherical harmonic — in particular, a harmonic homogeneous polynomial of degree n in the variables $\zeta_1, \zeta_2, \dots, \zeta_p$. Therefore, the function F is also a harmonic homogeneous polynomial in the components of ζ , i.e., in $s, \sqrt{1-s^2}$. But, since we equally well could have chosen coordinates such that $\zeta = (s, -\sqrt{1-s^2}, 0, \dots, 0)$, F must really be a polynomial in s .

Then, being only a function of the inner product $s = \langle \zeta, \xi \rangle$, we see that F satisfies all the defining properties of the Legendre polynomials in Theorem 4.10 except (i). We conclude

$$F(\zeta, \xi) = cP_n(s), \quad \text{for some constant } c.$$

We can determine the constant by considering the case $s = 1$, i.e., $\zeta = \xi$, where $F(\xi, \xi) = cP_n(1) = c$. Using Lemma 4.17,

$$c = \int_{\eta \in S^{p-1}} f(\langle \xi, \eta \rangle) P_n(\langle \eta, \xi \rangle) d\Omega_{p-1} = \Omega_{p-2} \int_{-1}^1 f(t) P_n(t) (1-t^2)^{(p-3)/2} dt,$$

as sought. ■

Theorem 4.24 (Hecke-Funk Theorem) *Let ξ be a unit vector, f a function, and Y_n a spherical harmonic. Then,*

$$\int_{\eta \in S^{p-1}} f(\langle \xi, \eta \rangle) Y_n(\eta) d\Omega_{p-1} = \Omega_{p-2} Y_n(\xi) \int_{-1}^1 f(t) P_n(t) (1-t)^{(p-3)/2} dt. \quad (4.36)$$

Proof By Theorem 4.13, there exist a set of coefficients $\{a_k\}_{k=1}^{N(p,n)}$ and a set of unit vectors $\{\zeta_k\}_{k=1}^{N(p,n)}$ such that

$$Y_n(\eta) = \sum_{k=1}^{N(p,n)} a_k P_n(\langle \eta, \zeta_k \rangle).$$

Then, we can rewrite the left-hand side of (4.36) as

$$\sum_{k=1}^{N(p,n)} a_k \int_{\eta \in S^{p-1}} f(\langle \xi, \eta \rangle) P_n(\langle \eta, \zeta_k \rangle) d\Omega_{p-1},$$

or, using Lemma 4.23,

$$\Omega_{p-2} \left(\sum_{k=1}^{N(p,n)} a_k P_n(\langle \xi, \zeta_k \rangle) \right) \int_{-1}^1 f(t) P_n(t) (1-t^2)^{(p-3)/2} dt.$$

That is

$$\Omega_{p-2} Y_n(\xi) \int_{-1}^1 f(t) P_n(t) (1-t^2)^{(p-3)/2} dt,$$

which is exactly the right-hand side of (4.36). ■

We wish to find now an integral representation of the Legendre polynomials. Towards this goal, we first prove a lemma.

Consider a vector η of the $(p-1)$ -dimensional sphere S^{p-1} . Without loss of generality, we may take it along the x_1 -axis. With $\{\eta\}^\perp$ we indicate the set of all vectors that are perpendicular to η ; this is obviously a hyperplane. The set $\{\eta\}^\perp \cap S^{p-1}$ is then the equator of S^{p-1} (a $(p-2)$ -sphere) that is orthogonal to η ; we will indicate it by S_η^{p-1} .

Lemma 4.25 *Let $\eta = (1, 0, \dots, 0)$ and $x \in \mathbb{R}^p$. Then the function*

$$L_n(x) = \frac{1}{\Omega_{p-2}} \int_{\zeta \in S_\eta^{p-1}} (\langle x, \eta \rangle + i \langle x, \zeta \rangle)^n d\Omega_{p-2}, \quad (4.37)$$

when restricted to the sphere, is the n -th Legendre polynomial: $L_n(\xi) = P_n(\langle \xi, \eta \rangle)$.

Proof Clearly, $L_n(x)$ is a polynomial in the components of x . By the binomial theorem, this polynomial is homogeneous of degree n . We can also see that $L_n(x)$ is harmonic. Indeed, applying the Laplace operator on (4.37) and switching the order of differentiation and integration, the integrand becomes

$$\sum_{j=1}^p \frac{\partial^2}{\partial x_j^2} (\langle x, \eta \rangle + i \langle x, \zeta \rangle)^n,$$

or

$$\sum_{j=1}^p \frac{\partial}{\partial x_j} \left[n (\langle x, \eta \rangle + i \langle x, \zeta \rangle)^{n-1} (\eta_j + i \zeta_j) \right],$$

or

$$n(n-1) (\langle x, \eta \rangle + i \langle x, \zeta \rangle)^{n-2} \sum_{j=1}^p (\eta_j + i \zeta_j)^2.$$

But

$$\sum_{j=1}^p (\eta_j + i \zeta_j)^2 = \langle \eta, \eta \rangle - \langle \zeta, \zeta \rangle + 2i \langle \eta, \zeta \rangle = 0,$$

since $\{\eta, \zeta\}$ is an orthonormal set. Also, notice

$$\begin{aligned} L_n(\eta) &= \frac{1}{\Omega_{p-2}} \int_{\zeta \in S_\eta^{p-1}} [\langle \eta, \eta \rangle + i \langle \eta, \zeta \rangle]^n d\Omega_{p-2} \\ &= \frac{1}{\Omega_{p-2}} \int_{\zeta \in S_\eta^{p-1}} d\Omega_{p-2} = 1. \end{aligned}$$

Now, let R be any coordinate rotation leaving η invariant, i.e., let R be any rotation about the x_1 -axis. The integrand of $L_n(Rx)$ is then

$$[\langle Rx, \eta \rangle + i \langle Rx, \zeta \rangle]^n = [\langle x, \eta \rangle + i \langle x, R^t \zeta \rangle]^n,$$

which can be reset to

$$[\langle x, \eta \rangle + i \langle x, \zeta \rangle]^n$$

by a change of variables $\zeta \rightarrow \zeta' = R\zeta$ and thus $L_n(x)$ is invariant under all such coordinate rotations.

We have shown that $L_n(x)$ has all the properties described in Theorem 4.10, so that, when restricted to the sphere, it becomes the n -th Legendre polynomial. ■

Now we can prove the following integral representation for $P_n(t)$.

Theorem 4.26

$$P_n(t) = \frac{\Omega_{p-3}}{\Omega_{p-2}} \int_{-1}^1 \left(t + is\sqrt{1-t^2} \right)^n (1-s^2)^{(p-4)/2} ds.$$

Proof From Lemma 4.37, we have

$$P_n(\langle \xi, \eta \rangle) = \frac{1}{\Omega_{p-2}} \int_{\zeta \in S_\eta^{p-1}} (\langle \xi, \eta \rangle + i \langle \xi, \zeta \rangle)^n d\Omega_{p-2}.$$

Choose a constant t and unit vector ν such that $\xi = t\eta + \sqrt{1-t^2}\nu$ and $\langle \nu, \eta \rangle = 0$. Then, the above equation becomes, using Lemma 4.17 and replacing p by $p-1$,

$$\begin{aligned} P_n(t) &= \frac{1}{\Omega_{p-2}} \int_{\zeta \in S_\eta^{p-1}} \left(t + i\sqrt{1-t^2} \langle \nu, \zeta \rangle \right)^n d\Omega_{p-2} \\ &= \frac{\Omega_{p-3}}{\Omega_{p-2}} \int_{-1}^1 \left(t + is\sqrt{1-t^2} \right)^n (1-s^2)^{(p-4)/2} ds, \end{aligned}$$

as sought. ■

4.4 Boundary Value Problems

We conclude this discussion with an application of the ideas we have developed to boundary value problems, where they display most of their physical importance.

We know from Proposition 3.15 that if a set of functions in a Hilbert space is closed, then it is complete. In the following theorem, we will see that a maximal linearly independent set of spherical harmonics of all degrees is closed and thus complete. This result allows us to develop expansions of functions as linear combinations of spherical harmonics, which will be useful in application to boundary value problems. In what follows, let

$$S = \{Y_{n,j} : n \in \mathbb{N}_0, 1 \leq j \leq N(p, n)\},$$

be a maximal set of orthogonal spherical harmonics, and let $\sum_{n,j}$ denote the sum $\sum_{n=0}^{\infty} \sum_{j=1}^{N(p,n)}$.

Theorem 4.27 *Let the function $f : S^{p-1} \rightarrow \mathbb{R}$ be continuous. If f is orthogonal to the set S , i.e., if*

$$\int_{\xi \in S^{p-1}} f(\xi) Y_{n,j}(\xi) d\Omega_{p-1} = 0, \text{ for all } n, j$$

then f is the zero function, i.e.,

$$f(\xi) = 0, \text{ for all } \xi \in S^{p-1}.$$

The requirement that f be continuous is actually too strict, and the theorem really applies to all square-integrable functions f , i.e., all f such that

$$\int_{S^{p-1}} f(\xi)^2 d\Omega_{p-1} < \infty.$$

However, we will only prove the weaker version of this theorem.

Proof We will prove it by contradiction. Suppose that f satisfies the hypotheses of the above theorem, i.e., is continuous and orthogonal to S , but is not the zero function. Then there exists some $\eta \in S^{p-1}$ such that $f(\eta) \neq 0$. We can assume that $f(\eta) > 0$, for if $f(\eta)$ is negative we could consider $-f$ instead. By the continuity of f , there is some neighborhood around η on the sphere where f is positive. That is, there exists a constant s such that $f(\xi) > 0$ whenever $s \leq \langle \xi, \eta \rangle \leq 1$. Define the function

$$\psi(t) = \begin{cases} 1 - \frac{(1-t)^2}{(1-s)^2} & \text{if } s \leq t \leq 1, \\ 0 & \text{if } -1 \leq t \leq s. \end{cases}$$

For $s \leq \langle \xi, \eta \rangle \leq 1$, the product $f(\xi)\psi(\langle \xi, \eta \rangle)$ is positive, and it vanishes for all other ξ . Thus,

$$\int_{\xi \in S^{p-1}} f(\xi)\psi(\langle \xi, \eta \rangle) d\Omega_{p-1} c > 0. \quad (4.38)$$

For simplicity, we will indicate the above integral by c . By the Weierstrass approximation theorem (Proposition 3.10), we can find a polynomial $p(t)$ for any given $\epsilon > 0$ such that

$$|\psi(t) - p(t)| \leq \epsilon, \text{ for all } t \in [-1, 1].$$

For any such ϵ and $p(t)$,

$$\begin{aligned} \int_{\xi \in S^{p-1}} f(\xi) [\psi(\langle \xi, \eta \rangle) - p(\langle \xi, \eta \rangle)] d\Omega_{p-1} &\leq \left| \int_{\xi \in S^{p-1}} f(\xi) [\psi(\langle \xi, \eta \rangle) - p(\langle \xi, \eta \rangle)] d\Omega_{p-1} \right| \\ &\leq \int_{\xi \in S^{p-1}} |f(\xi)| |\psi(\langle \xi, \eta \rangle) - p(\langle \xi, \eta \rangle)| d\Omega_{p-1} \\ &\leq \epsilon \int_{\xi \in S^{p-1}} |f(\xi)| d\Omega_{p-1}. \end{aligned}$$

Since $f(\xi)$ is continuous and $\xi \in S^{p-1}$, there exists an M such that $M \geq |f(\xi)|$ for any $\xi \in S^{p-1}$. This implies

$$\int_{\xi \in S^{p-1}} f(\xi)p(\langle \xi, \eta \rangle) d\Omega_{p-1} \geq c - M\epsilon \Omega_{p-1}.$$

And since we can choose ϵ arbitrarily small,

$$\int_{\xi \in S^{p-1}} f(\xi) p(\langle \xi, \eta \rangle) d\Omega_{p-1} > 0. \quad (4.39)$$

For the remainder of the proof, we fix ϵ and the corresponding $p(t)$ for which this expression is true.

Now, let m denote the degree of $p(t)$. We can write the polynomial $p(t)$ as a linear combination of the first m Legendre polynomials since each $P_n(t)$ is of degree n ; i.e., we can find c_k such that

$$p(t) = \sum_{k=0}^m c_k P_k(t).$$

We can thus rewrite the integral in (4.39) as

$$\int_{\xi \in S^{p-1}} f(\xi) \sum_{k=0}^m c_k P_k(\langle \xi, \eta \rangle) d\Omega_{p-1}.$$

But since the Legendre polynomials are just a special collection of spherical harmonics in ξ , this integral must vanish by hypothesis, contradicting the assertion in (4.39). Therefore, our initial assumption that f is not the zero function must be false. ■

So for any reasonable function f defined on the sphere, we can write

$$f(\xi) = \sum_{n,j} c_{n,j} Y_{n,j}(\xi). \quad (4.40)$$

To find $c_{n',j'}$, multiply both sides of this equation by $Y_{n',j'}$ and integrate over the sphere. Using the orthonormality of the set S , we find

$$c_{n',j'} = \int_{S^{p-1}} f(\xi) Y_{n',j'}(\xi) d\Omega_{p-1}. \quad (4.41)$$

This expansion will be used in the following demonstration.

Problem Consider the following boundary-value problem. Find V in the closed unit ball \bar{B}^p , such that

$$\Delta_p V = 0, \quad \text{and} \quad V = f(\xi), \quad \text{for all } \xi \in S^{p-1}. \quad (4.42)$$

Solution We know that harmonic homogeneous polynomials are solutions to the Laplace equation, as well as any linear combination of them. We have also seen in (4.7) that we can write each of these polynomials as a power of the radius multiplied by a spherical harmonic. Thus, we can construct the solution to (4.42) as a linear combination of $r^n Y_{n,j}(\xi)$ terms. To satisfy the boundary condition, we can use the coefficients in (4.41) to find

$$V = \sum_{n,j} r^n c_{n,j} Y_{n,j}(\xi) = \sum_{n,j} r^n Y_{n,j}(\xi) \int_{S^{p-1}} f(\eta) Y_{n,j}(\eta) d\Omega_{p-1}, \quad (4.43)$$

thus solving the problem with the solution being in the form of a series. ■

In the next subsection, we will learn to solve this problem by another method. Since the solution of this boundary-value problem is unique (as we know from the theory of differential equations), we can equate the answers. And, amazingly, this procedure will give us a generating function for the Legendre polynomials $P_n(t)$.

Green's Functions

Given a differential equation

$$D_x(f(x)) = 0,$$

where D_x is a differential operator acting on the unknown function $f(x)$, the corresponding Green's function G is defined by the equation

$$D_x(G(x)) = \delta(x - x_0),$$

where the function δ appearing in the right-hand side is the Dirac delta function defined by

$$\delta(x - x_0) = 0, \text{ for all } x \neq x_0,$$

and

$$\int_{B_\epsilon^p(x_0)} \delta(x - x_0) d^p x = 1, \text{ for all } \epsilon > 0.$$

Let's assume that the given differential operator is the Laplacian in p dimensions, that is, we seek the Green's function which satisfies

$$\Delta_p \tilde{G} = \delta(x - x_0). \quad (4.44)$$

If we think of \tilde{G} as electric potential, then (4.44) describes the electric potential caused by a point charge at $x_0 \in \mathbb{R}^p$.

Since the Laplacian is invariant under translations, we see that \tilde{G} can only depend on the distance from x_0 , i.e., on $\rho = |x - x_0|$. When $\rho \neq 0$, \tilde{G} must satisfy Laplace's equation:

$$\begin{aligned}
 0 &= \Delta_p \tilde{G}(\rho) = \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \tilde{G}(\rho) \\
 &= \sum_{i=1}^p \frac{\partial}{\partial x_i} \left(\frac{\partial \tilde{G}(\rho)}{\partial \rho} \frac{\partial \rho}{\partial x_i} \right) \\
 &= \tilde{G}''(\rho) \sum_{i=1}^p \left(\frac{\partial \rho}{\partial x_i} \right)^2 + \tilde{G}'(\rho) \sum_{i=1}^p \frac{\partial^2 \rho}{\partial x_i^2} \\
 &= \tilde{G}''(\rho) + \frac{p-1}{\rho} \tilde{G}'(\rho).
 \end{aligned}$$

We can easily solve this differential equation by separation of variables to find

$$G(\rho) = a\rho + b, \quad \text{if } p = 1,$$

and

$$\tilde{G}'(\rho) = \frac{\text{const.}}{\rho^{p-1}},$$

if $p \geq 2$. This, in turn, implies

$$\tilde{G}(\rho) = a \ln \rho, \quad \text{if } p = 2,$$

and

$$\tilde{G}(\rho) = \frac{a}{\rho^{p-2}}, \quad \text{if } p \geq 3.$$

We will focus on the last case but the reader should explore the cases $p = 1, 2$ on his or her own to get a better understanding. To find the undetermined constant, we integrate the defining equation over a ball of radius ϵ centered at the point x_0 :

$$\int_{B_\epsilon^p(x_0)} \Delta_p \tilde{G} d^p x = \int_{B_\epsilon^p(x_0)} \delta(x - x_0) d^p x.$$

By the properties of the Dirac delta function, the right-hand side is 1. The

left-hand side can be rewritten by the use of the divergence theorem:

$$\begin{aligned}
 \int_{B_\epsilon^p(x_0)} \Delta_p \tilde{G} d^p x &= \int_{S_\epsilon^{p-1}(x_0)} \nabla_p \tilde{G} \cdot \xi (\epsilon^{p-1} d\Omega_{p-1}) \\
 &= \epsilon^{p-1} \int_{S_\epsilon^{p-1}(x_0)} \frac{d\tilde{G}}{d\rho} d\Omega_{p-1} \\
 &= \epsilon^{p-1} \int_{S_\epsilon^{p-1}(x_0)} \frac{a(2-p)}{\epsilon^{p-1}} d\Omega_{p-1} \\
 &= a(2-p) \Omega_{p-1}.
 \end{aligned}$$

Hence

$$a = \frac{1}{(2-p) \Omega_{p-1}}.$$

Of course, differential equations come with boundary conditions. So, let's modify the previous problem as follows. Let's seek the Green's function which satisfies the same equation

$$\Delta_p G = \delta(x - x_0), \quad (4.45)$$

for all $x \in B_p(0)$ and subjected to the boundary condition

$$G(\xi) = 0, \text{ for all } \xi \in S^{p-1}. \quad (4.46)$$

Equation (4.45) now describes the electric potential caused by a point charge at x_0 and an ideal conducting sphere with center at the origin.

To construct G , we write

$$G(x; x_0) = \tilde{G}(\rho) + g = \frac{1}{(2-p)\Omega_{p-1}\rho^{p-2}} + g,$$

and require that g is harmonic in B_p

$$\Delta_p g = 0$$

and cancel \tilde{G} on the boundary of B_p :

$$g(\xi) = -\tilde{G}(\xi) = \frac{1}{(p-2)\Omega_{p-1}\rho^{p-2}} \text{ for all } \xi \in S^{p-1}.$$

In fact, the functional expression of g is identical to that of \tilde{G} . However, there are two parameters that have to be fixed: the location of the singular point representing a point charge and the strength of the charge. The location of

the singular point of g cannot be inside $B_p(0)$. We will place the charge at the symmetric point x'_0 to x_0 with respect to the sphere, i.e., the point x'_0 that lies on the line passing through the origin and x_0 with $|x'_0| = 1/|x_0|$. We see then that

$$g(\rho') \propto \frac{1}{(2-p)\Omega_{p-1}\rho'^{p-2}},$$

where $\rho' = |x - x'_0|$. We must choose the charge to be of the correct strength so that G vanishes on the sphere. We easily see that the correct choice of g is

$$g(\rho') = -\frac{1}{(2-p)\Omega_{p-1}(|x_0|\rho')^{p-2}},$$

so that

$$G(x; x_0) = \frac{1}{(2-p)\Omega_{p-1}} \left(\frac{1}{\rho^{p-2}} - \frac{1}{(|x_0|\rho')^{p-2}} \right). \quad (4.47)$$

Indeed, (4.47) satisfies Laplace's equation. Using the law of cosines and letting θ be the angle between the ray from the origin to x and the ray from the origin to x_0 ,

$$\begin{aligned} \rho &= \sqrt{|x|^2 + |x_0|^2 - 2|x||x_0|\cos\theta}, \\ \rho' &= \sqrt{|x|^2 + |x'_0|^2 - 2|x||x'_0|\cos\theta} = \sqrt{|x|^2 + \frac{1}{|x_0|^2} - 2\frac{|x|}{|x_0|}\cos\theta}, \end{aligned}$$

we see that (4.47) vanishes on the unit sphere, i.e., when $|x| = 1$. The method of constructing G as described above is known as *the method of images*. Physicists use it routinely without paying attention to the mathematical details! The reason is that if a solution is found for a boundary problem, it must be unique.

Let's now return to the problem stated on page 79 and present an alternative solution using the results on the Green's function for the Laplace equation.

Alternative Solution Green's theorem (as shown in the footnote of page 55) for the function G and V ,

$$\int_{B^p} (V\Delta_p G - G\Delta_p V) d^p x = \int_{S^{p-1}} \left(V \frac{\partial G}{\partial \xi} - G \frac{\partial V}{\partial \xi} \right) d\Omega_{p-1},$$

reduces to

$$\int_{B^p} V \delta(x - x_0) d^p x = \int_{S^{p-1}} V \frac{\partial G}{\partial \xi} d\Omega_{p-1},$$

since V is harmonic and G obeys (4.45) and (4.46). Since the left-hand side of the above equation becomes

$$\int_{B^p} V \delta(x - x_0) d^p x = V(x_0) \int_{B^p} \delta(x - x_0) = V(x_0),$$

and since

$$\frac{\partial G}{\partial \xi} = \frac{\partial G}{\partial |x|} \Big|_{|x|=1} = \frac{1 - |x_0|^2}{\Omega_{p-1}(1 + |x_0|^2 - 2|x_0| \cos \theta)^{p/2}},$$

we can write the solution,

$$V(x_0) = \frac{1}{\Omega_{p-1}} \int_{\xi \in S^{p-1}} f(\xi) \frac{1 - |x_0|^2}{(1 + |x_0|^2 - 2|x_0| \cos \theta)^{p/2}} d\Omega_{p-1}. \quad (4.48)$$

Obviously, this gives the potential as an integral representation. ■

Equating the two solutions, we can arrive at the following result.

Theorem 4.28 *In \mathbb{R}^p we have*

$$\sum_{n=0}^{\infty} r^n N(p, n) P_n(t) = \frac{1 - r^2}{(1 - 2rt + r^2)^{p/2}}.$$

Proof Let $x_0 = |x_0|\eta$. Starting from equation (4.43),

$$\begin{aligned} V(x_0) &= \sum_{n,j} |x_0|^n Y_{n,j}(\eta) \int_{\xi \in S^{p-1}} f(\xi) Y_{n,j}(\xi) d\Omega_{p-1} \\ &= \sum_{n=0}^{\infty} |x_0|^n \int_{\xi \in S^{p-1}} f(\xi) \sum_{j=1}^{N(p,n)} Y_{n,j}(\xi) Y_{n,j}(\eta) d\Omega_{p-1}, \end{aligned}$$

we rewrite it using Theorem 4.11,

$$\begin{aligned} V(x_0) &= \sum_{n=0}^{\infty} |x_0|^n \int_{\xi \in S^{p-1}} f(\xi) \frac{N(p, n)}{\Omega_{p-1}} P_n(\langle \xi, \eta \rangle) d\Omega_{p-1} \\ &= \frac{1}{\Omega_{p-1}} \int_{\xi \in S^{p-1}} f(\xi) \sum_{n=0}^{\infty} |x_0|^n N(p, n) P_n(\cos \theta) d\Omega_{p-1}. \end{aligned}$$

Since the function f is arbitrary, we can compare the above equation with (4.48) and set $r = |x_0|$ and $t = \cos \theta$ to complete the proof. ■

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This concludes our development of spherical harmonics in p dimensions. We have briefly considered an application to boundary value problems; we will not delve further into applications. We have achieved our main goal to study the theory of spherical harmonics and the corresponding Legendre polynomials in \mathbb{R}^p . We urge the reader to seek out applications on his or her own. Perhaps search for instances in physics where spherical harmonics are used in \mathbb{R}^3 and try to generalize to p dimensions. One could start with the multipole expansion of an electrostatic field (see [6], [7]) or the wave function of an electron in a hydrogenic atom (see [11], [8]).

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